The Circular Extremality of Some Reduced Kneser Graphs

Wei-Tian Li
Department of Mathematics
National Taiwan University
Taipei 106, Taiwan

Ko-Wei Lih
Institute of Mathematics
Academia Sinica
Taipei 115, Taiwan

June 2, 2003

Abstract

For $m \geq 2n$, the *Kneser graph* $\mathsf{KG}(m,n)$ has the collection of all n-subsets of $[m] = \{0, 1, \ldots, m-1\}$ as its vertex set, and two vertices are adjacent if and only if they are disjoint as subsets. A subset of $[m]$ is called 2-stable if $2 \leq |u - v| \leq m - 2$ for distinct elements u and v. The reduced Kneser graph $\mathsf{KG}_2(m,n)$ is the subgraph of $\mathsf{KG}(m,n)$ induced by 2-stable subsets. We prove that the circular chromatic number of $\mathsf{KG}_2(2n+2,n)$ is equal to its chromatic number.

Keywords: circular chromatic number, orientation, flow ratio, reduced Kneser graph

1 Introduction

In 1988, Vince [8] introduced a generalized notion for graph coloring. What he called the star chromatic number is now widely known as the circular chromatic number. Let k and d be positive integers such that $k \geq 2d$. A (k,d)-coloring of a graph G is a mapping σ from the vertex set of G to the set $[k] = \{0, 1, \ldots, k-1\}$ such that $d \leq |\sigma(u) - \sigma(v)| \leq k - d$ for any two adjacent vertices u and v. We may define the *k-circular distance* $|x - y|_k$ between x and

y to be $\min\{|x-y|, k-|x-y|\}$. Thus, a (k,d)-coloring requires two adjacent vertices be mapped to numbers whose k-circular distance is at least d. A $(k,1)$-coloring is exactly an ordinary proper coloring. The *circular chromatic number* $\chi^\circ(G)$ of G is defined to be the infimum of k/d such that there exists a (k,d)-coloring of G. It is well-known that $\chi(G)-1 < \chi^\circ(G) \le \chi(G)$, where $\chi(G)$ denotes the ordinary chromatic number of the graph G.

Zhu [10] provides a comprehensive survey of recent research on circular chromatic numbers. Actually, there are several equivalent ways of defining the circular chromatic number. One approach that is particular useful in this paper is through flow ratios.

Let G be a graph endowed with an orientation D, i.e., each edge is given a direction so that it becomes an arc. We call such a graph G an *oriented* graph and denote it by $(G;D)$. Let C be a cycle of the underlying graph G. When we move around C, there are two possible choices for the direction of traversal. Let us fix one for reference. An arc of C is called *forward* if its orientation under D coincides with the direction of traversal; otherwise it is called *backward*. We denote the number of forward arcs of C by C^+ and the number of backward arcs of C by C^-. The *flow ratio* $\theta(C;D)$ of C under the orientation D is defined to be $\max\{C^+/C^-, C^-/C^+\}$. This quantity measures the extent of imbalance between arcs in the two opposite directions, and it is independent of the choice of the direction of traversal. We allow the flow ratio to be ∞ when C is a directed cycle under D. The *flow ratio* $\theta(G;D)$ of an oriented graph $(G;D)$ is defined to be the number $\max\{\theta(C;D) \mid C \text{ is a cycle of } G.\}$ We stipulate that $\theta(G;D)$ is 1 when G is a forest.

The following theorem was proved by Goddyn, Tarsi, and Zhang [1] using tools from matroid theory. It furnishes us with an equivalent definition for the circular chromatic number.

Theorem 1 *The circular chromatic number $\chi^\circ(G)$ of a graph G satisfies the following identity.*

$$\chi^\circ(G) = 1 + \min\{\theta(G;D) \mid D \textit{ is an orientation of } G.\}$$

For $m \geq 2n$, the *Kneser graph* $\mathsf{KG}(m,n)$ has the collection of all n-subsets of $[m]$ as its vertex set, and two vertices are adjacent if and only if they are disjoint as subsets. The following was conjectured by Kneser [4] in 1955 and established by Lovász [6] in 1978.

Theorem 2 *For every Kneser graph $\mathsf{KG}(m,n)$, we have $\chi(\mathsf{KG}(m,n)) = m - 2n + 2$.*

In 1997, Johnson, Holroyd, and Stahl [3] determined the following circular chromatic numbers of Kneser graphs: $\chi^\circ(\mathsf{KG}(2n+1,n)) = 3$, $\chi^\circ(\mathsf{KG}(2n+2,n)) = 4$, and $\chi^\circ(\mathsf{KG}(m,2)) = m-2$. We call a graph G *circular extremal* if $\chi^\circ(G) = \chi(G)$. They also proposed the following.

Conjecture 1 *Every Kneser graph $\mathsf{KG}(m,n)$ is circular extremal.*

A subset S of $[m]$ is said to be *2-stable* if $|u-v|_m \geq 2$ for any two distinct elements u and v of S, or equivalently, S contains no two consecutive numbers in the cyclic order of $[m]$. The *reduced Kneser graph* $\mathsf{KG}_2(m,n)$ is the subgraph of $\mathsf{KG}(m,n)$ induced by all vertices that are 2-stable subsets. Schrijver [7] proved that $\chi(\mathsf{KG}_2(m,n)) = \chi(\mathsf{KG}(m,n))$ and $\mathsf{KG}_2(m,n)$ is vertex color critical. Lih and Liu [5] established the following.

Theorem 3 *The reduced Kneser graph $\mathsf{KG}_2(m,2)$ is circular extremal when $m \geq 4$ and $m \neq 5$.*

It is easy to see that $\mathsf{KG}_2(2n+1,n)$ is the odd cycle C_{2n+1}. Therefore $\chi^\circ(\mathsf{KG}_2(2n+1,n)) = 2+\frac{1}{n}$ and $\mathsf{KG}_2(2n+1,n)$ fails to be circular extremal. Let $t(n)$ denote the least number n_0 such that $\mathsf{KG}_2(m,n)$ is circular extremal if $m \geq n_0$. The existence of $t(n)$ was posed as a problem in Lih and Liu [5]. Recently, Hajiabolhassan and Zhu [2] answered it in the affirmative. Their result implies that Conjecture 1 is true for sufficiently large m for any given n. The determination of the precise values of all $t(n)$ remains open, except $t(2) = 6$ and $t(n) > 2n+1$.

In this paper, we will show that $\mathsf{KG}_2(2n+2,n)$ is circular extremal. This does not necessarily imply that $t(n) = 2n+2$. However, if it turns out that $t(n) = 2n+2$, then Conjecture 1 will be its immediate consequence.

2 The Structure of $\mathsf{KG}_2(2n+2,n)$

Let S be a 2-stable n-subset of $[m]$ containing a fixed number x. Starting from x, we enumerate the m-circular distances between consecutive elements of S in the cyclic order. The sequence of "gaps" $x_i \geq 2$, $1 \leq i \leq n$, satisfy the equation $x_1 + x_2 + \ldots + x_n = m$. Conversely, any such sequence of integers determine a 2-stable n-subset of $[m]$ containing x. Therefore, The vertex set of the reduced Kneser graph $\mathsf{KG}_2(m,n)$ has size $\frac{m}{n}\binom{m-n-1}{n-1}$.

When $m = 2n+2$ and $n \geq 2$, there are two types of gaps that satisfy the above-mentioned requirements. In the first type, all gaps are equal to 2, except one gap is equal to 4. In the second type, all gaps are equal to 2, except two gaps are equal to 3. A vertex of $\mathsf{KG}_2(2n+2,n)$ is classified as *type-1* if it has the form

$$u(i) = \{i, i+4, i+6, \ldots, i+2n\},$$

where $0 \le i < 2n+2$, as *type-2* if it has the form

$$v(i,j) = \{i, i+3, i+5, \ldots, j, j+3, j+5, \ldots, i+2n\},$$

where $0 \le i, j < 2n+2$ and $|i-j|_{2n+2} \ge 3$. Here and henceforth, expressions for elements in a vertex is understood modulo $2n+2$. Due to the cyclic order of the elements of $[2n+2]$, both $v(i,j)$ and $v(j,i)$ denote the same vertex.

Let U denote the set of all type-1 vertices. The size of U is precisely $2n+2$. Let V denote the set of all type-2 vertices. Then V can be partitioned into the sets $V_d = \{v(i,j) \mid |i-j|_{2n+2} = d\}$. We observe that the index d of V_d is an odd integer that is at least 3 and at most n for odd n, or $n+1$ for even n. The size of V_d is precisely $2n+2$, except V_{n+1} has only $n+1$ vertices when n is even.

Let us examine the adjacencies among vertices of $\mathsf{KG}_2(2n+2,n)$. We see that $u(i)$ is adjacent to exactly the following $n+2$ vertices: $v(i-1,i+2)$ and $u(i+2j+1)$ for $0 \le j \le n$.

Let S be a subset of $[m] \setminus \{0\}$. The *circulant graph* $C(m;S)$ is the graph on the vertex set $\{v_0, v_1, \ldots, v_{m-1}\}$ such that v_i and v_j are adjacent if and only if $|i-j|_m \in S$. If $1 \in S$, then the cycle $v_0 v_1 \cdots v_{m-1} v_0$ is called the *boundary* of $C(m;S)$. Let $[m]_{odd}$ denote the subset of $[m]$ consisting of all odd numbers. Thus the set U induces a subgraph of $\mathsf{KG}_2(2n+2,n)$ isomorphic to the circulant graph $C(2n+2,[n+1]_{odd})$, which is in turn isomorphic to the complete bipartite graph $K_{n+1,n+1}$.

A vertex $v(i,j)$ in V_d has precisely four neighbors: $v(i-1,j-1)$ and $v(i+1,j+1)$ in V_d, $v(i+1,j-1)$ in V_{d-2}, and $v(i-1,j+1)$ in V_{d+2}, subject to the following modifications. (1) When $j = i+3$, the vertex $v(i+1,j-1)$ should be replace by $u(i+1)$ in U. (2) When n is odd and $j = i+n$, we see that $v(i-1,j+1) = v(j+1,i-1) = v(i+n+1,i+2n+1)$ is also in V_n.

The *Cartesian product* $G_1 \Box G_2$ of two graphs G_1 and G_2 is the graph G

on the Cartesian product of the two vertex sets so that two vertices (u_1, v_1) and (u_2, v_2) of G are adjacent if u_1 is adjacent to u_2 and $v_1 = v_2$, or $u_1 = u_2$ and v_1 is adjacent to v_2. The Cartesian product of the path P_{m+1} of length m with the path P_{n+1} of length n is called a *grid graph* $\mathsf{G}(m, n)$, and its vertices are denoted by $g(i, j)$ for $0 \le i \le m$ and $0 \le j \le n$.

By a homomorphism between two graphs, we mean an adjacency preserving mapping between the graphs. We are going to show that the following mapping τ is a surjective homomorphism from $\mathsf{G}(n+1, n)$ to $\mathsf{KG}_2(2n+2, n)$.

$$\tau(g(i,j)) = \begin{cases} u(\lfloor \frac{3n+2}{2} \rfloor + i) & \text{if } j = 0, \\ u(\lfloor \frac{n}{2} \rfloor + i) & \text{if } j = n, \\ v(\lfloor \frac{3n+4}{2} \rfloor + i + j, \lfloor \frac{3n+2}{2} \rfloor + i - j) & \text{if } 0 < j < n. \end{cases}$$

We see that the sequence $\tau(g(0,0))$, $\tau(g(1,0))$, $\ldots$, $\tau(g(n+1,0))$, $\tau(g(0,n))$, $\tau(g(1,n))$, $\ldots$, $\tau(g(n+1,n))$ runs through the vertices of U exactly once, except $\tau(g(n+1,0)) = u(\lfloor \frac{5n+4}{2} \rfloor) = u(\lfloor \frac{n}{2} \rfloor) = \tau(g(0,n))$ and $\tau(g(n+1,n)) = u(\lfloor \frac{3n+2}{2} \rfloor) = \tau(g(0,0))$.

When $0 < j < n$ and $j \neq \frac{n}{2}$, we see that the sequence $\tau(g(0,j))$, $\tau(g(1,j))$, $\ldots$, $\tau(g(n+1,j))$, $\tau(g(0,n-j))$, $\tau(g(1,n-j))$, $\ldots$, $\tau(g(n+1,n-j))$ runs through the vertices of V_{2j+1} exactly once, except $\tau(g(n+1,j)) = v(\lfloor \frac{3n+4}{2} \rfloor + n+1+j, \lfloor \frac{3n+2}{2} \rfloor + n+1-j) = v(\lfloor \frac{3n+4}{2} \rfloor + n-j, \lfloor \frac{3n+2}{2} \rfloor - n+j) = \tau(g(0,n-j))$ and $\tau(g(n+1,n-j)) = v(\lfloor \frac{3n+4}{2} \rfloor + 2n+1-j, \lfloor \frac{3n+2}{2} \rfloor + 1+j) = v(\lfloor \frac{3n+4}{2} \rfloor + j, \lfloor \frac{3n+2}{2} \rfloor - j) = \tau(g(0,j))$. Furthermore, when $j = \frac{n}{2}$ and n is even, the sequence $\tau(g(0,j))$, $\tau(g(1,j))$, $\ldots$, $\tau(g(n+1,j))$ runs through the vertices of V_{n+1} exactly once, except $\tau(g(n+1,j)) = v(n+1,0) = v(0,n+1) = \tau(g(0,j))$.

We observe that $\tau(g(i,0)) = u(\lfloor \frac{3n+2}{2} \rfloor + i)$ is adjacent to $v(\lfloor \frac{3n}{2} \rfloor + i, \lfloor \frac{3n+6}{2} \rfloor + i) = v(\lfloor \frac{3n+6}{2} \rfloor + i, \lfloor \frac{3n}{2} \rfloor + i) = \tau(g(i,1))$ and $\tau(g(i,n)) = u(\lfloor \frac{n}{2} \rfloor + i)$ is adjacent to $v(\lfloor \frac{n-2}{2} \rfloor + i, \lfloor \frac{n+4}{2} \rfloor + i) = v(\lfloor \frac{3n+4}{2} \rfloor + i + n - 1, \lfloor \frac{3n+2}{2} \rfloor + i - n + 1) = \tau(g(i,n))$.

Now it is straightforward to see that τ is a surjective homomorphism from $\mathsf{G}(n+1,n)$ to $\mathsf{KG}_2(2n+2,n)$.

The *Möbius grid* $\mathsf{MG}(m,n)$ is obtained from $\mathsf{G}(m+1,n)$ by identifying the vertices $g(0,i)$ with the vertices $g(m+1,n-i)$ for $0 \leq i \leq n$. We still use $g(i,j)$ to denote the vertices of that Möbius grid. By the *boundary* of the Möbius grid $\mathsf{MG}(m,n)$, we mean the cycle $g(0,0)g(1,0)\cdots g(m+1,0)g(1,n)\cdots g(m+1,n)$. It can be checked immediately that τ is an isomorphism from $\mathsf{MG}(n,n)$ onto the subgraph of $\mathsf{KG}_2(2n+2,n)$ that is obtained by deleting every edge $u(i)u(j)$ such that $|i-j|_{2n+2} > 1$.

Let us summarize the analysis of the structure of $\mathsf{KG}_2(2n+2,n)$ into the following.

Theorem 4 *Let $n \geq 2$. If we collect all vertices $u(i) = \{i, i+4, i+6, \ldots, i+2n\}$, $0 \leq i < 2n+2$, of $\mathsf{KG}_2(2n+2,n)$ into the set U, then U induces the circulant graph $C(2n+2,[n+1]_{odd})$. If we delete every edge $u(i)u(j)$ such that $|i-j|_{2n+2} > 1$, then the resulting subgraph is isomorphic to the Möbius grid $\mathsf{MG}(n,n)$. Furthermore, the boundary of $C(2n+2,[n+1]_{odd})$ coincides with the boundary of $\mathsf{MG}(n,n)$.*

3 The Main Theorem

Lemma 5 *Suppose that every 4-cycle of the grid graph $\mathsf{G}(m,n)$ has flow ratio 1 under an orientation D. Then $\theta(C;D) = 1$ for every cycle C of the underlying grid graph.*

Proof. For every 4-cycles $g(i,j)g(i+1,j)g(i+1,j+1)g(i,j+1)g(i,j)$, $0 \leq i < m$ and $0 \leq j < n$, the traversal is chosen to move in the direction from $g(i,j)$ to $g(i+1,j)$. Then every arc shared by two such 4-cycles is forward in one cycle and backward in the other. The grid graph $\mathsf{G}(m,n)$ can be drawn in the Euclidean plane in a canonical way. Let C be a cycle of $\mathsf{G}(m,n)$. The traversal of C starts from the vertex satisfying the following properties. (1) It has the least first coordinate. (2) Among such vertices, it has the least second coordinate. Let this vertex be $g(i,j)$. Then the traversal is chosen to move in the direction from $g(i,j)$ to $g(i+1,j)$.

If two cycles share some common arcs, we say that the separate traversals of the two cycles are *consistent* when no arc is forward in one traversal and backward in the other. The region enclosed within C consists of a number of 4-cycles. Thus the traversal defined above is consistent with the traversal of every 4-cycles enclosed within C. When we count the number of forward, respectively backward, arcs of 4-cycles enclosed within C, each arc not included in C will be counted once as forward and once as backward. Since each 4-cycle contains exactly two forward and two backward arcs, the total numbers of forward and backward arcs of all the 4-cycles enclosed within C are identical. It implies that the number of forward and backward arcs on C are identical, too. Therefore $\theta(C;D)=1$. □

Corollary 6 *Let G be the circulant graph $C(2n+2;[n+1]_{odd})$. Suppose that every 4-cycle of G has flow ratio 1 under an orientation D. Then the boundary of G also has flow ratio 1 under D.*

Proof. Since v_iv_{2n+1-i} is an edge for all $0 \le i \le n$, we observe that these edges and the boundary of G form the grid graph $\mathsf{G}(n,1)$. Every 4-cycle of $\mathsf{G}(n,1)$ is a 4-cycle of G. Hence each 4-cycle of $G(n,1)$ has two forward and two backward arcs under D. It follows that the flow ratio of the boundary of the grid graph, which coincides with the boundary of $C(2n+2;[n+1]_{odd})$, is equal to 1. □

Lemma 7 *Suppose that every* 4*-cycle of the Möbius grid* $\mathsf{MG}(m,n)$ *has flow ratio 1 under an orientation* D*. If the parities of* m *and* n *are identical, then the flow ratio of its boundary under the orientation* D *is greater than 1.*

Proof. For every 4-cycle $g(i,j)g(i+1,j)g(i+1,j+1)g(i,j+1)g(i,j)$, $0 \le i \le m$ and $0 \le j < n$ in the grid graph $\mathsf{G}(m{+}1,n)$, the traversal is chosen to move in the direction from (i,j) to $(i+1,j)$. We observe that two such traversals are consistent on each identified edge $g(m{+}1,n{-}j)g(m{+}1,n{-}j{+}1)$ and $(0,j{+}1)(0,j)$ in $\mathsf{MG}(m,n)$. Thus the orientation D of $\mathsf{MG}(m,n)$ provided by the assumption gives rise to an orientation of $\mathsf{G}(m{+}1,n)$, and every 4-cycle of $\mathsf{G}(m+1,n)$ has two forward and two backward arcs under this orientation. Furthermore, if there are a forward and b backward arcs along the path $g(0,0)g(0,1)\cdots g(0,n)$, then there are a forward and b backward arcs along the path $g(m+1,0)g(m+1,1)\cdots g(m+1,n)$. Let there be c forward and d backward arcs along the path $g(0,0)g(1,0)\cdots g(m+1,0)$, and x forward and y backward arcs along the opposite path $P : g(0,n)g(1,n)\cdots g(m+1,n)$. Since the total number of forward arcs of $\mathsf{G}(m+1,n)$ is equal to the total number of backward arcs, it follows that the quantities a, b, c, d, x, and y satisfy the following equations.

$$\begin{aligned} x + y &= c + d \\ c + x + 2a &= d + y + 2b \end{aligned}$$

Solving these equations for x and y, we get $x = d + b - a$, $y = c - b + a$.

Now the traversal of the boundary of the Möbius grid is chosen to move in the direction from $g(0,0)$ to $g(1,0)$. If an arc is shared by a 4-cycle and the path P, then its direction of traversal in the 4-cycle is opposite to that in the boundary. Hence, there are $y + c = 2c - b + a$ forward and $x + d = 2d + b - a$ backward arcs on the boundary of the Möbius grid. Since $a + b = n$ and

$c + d = m + 1$ and the parities of $m + 1$ and n are different, it follows that the parities of $b - a$ and $c - d$ are also different. Consequently, $x + d \neq y + c$ and the flow ratio of the boundary of the Möbius grid is greater than one. $\Box$

Theorem 8 *The reduced Kneser graphs* $\mathsf{KG}_2(2n+2, n)$ *are circular extremal for all* n*.*

Proof. We want to show $\chi^\circ(\mathsf{KG}_2(2n+2, n)) = 4$. For $n = 1$, $\mathsf{KG}_2(2n+2, n))$ is K_4 whose circular chromatic number is 4. When $n \geq 2$, suppose that the circular chromatic number is less than 4. By Theorem 1, there exists an orientation D such that $\theta(\mathsf{KG}_2(2n + 2, n); D) < 3$. It follows that every 4-cycle of $\mathsf{KG}_2(2n + 2, n)$ has two forward and two backward arcs under D. From the structure of $\mathsf{KG}_2(2n + 2, n)$ given by Theorem 4, together with Corollary 6 and Lemma 7, we know that the flow ratio of the boundary of the circulant graph induced by the set U of type-1 vertices is simultaneously equal to and greater than 1. A contradiction is thus obtained. $\Box$

Since the reduced Kneser subgraph is an induced subgraph of the Kneser graph, we immediately get the following corollary, which was first obtained in Johnson, Holroyd, and Stahl [3] by a different approach.

Corollary 9 *The Kneser graphs* $\mathsf{KG}(2n + 2, n)$ *are circular extremal for all* n*.*

4 Remarks

We know from Zhu [9] that $\chi^\circ(G \Box H) = \max\{\chi^\circ(G), \chi^\circ(H)\}$. It follows that $\chi^\circ(G(m, n)) = \chi^\circ(P_{m+1} \Box P_{n+1}) = 2$. Therefore the assumption of Lemma 5 can be met. The next theorem shows that the assumption of Lemma 7 can also be satisfied.

Theorem 10 *The circular chromatic number of a Möbius grid is determined as follows.*

(1) $\chi^\circ(\mathsf{MG}(m,n)) = 2$ *if the parities of* m *and* n *are different.*

(2) $\chi^\circ(\mathsf{MG}(m,n)) = 2 + \frac{2}{m}$ *if the parities of* m *and* n *are identical.*

Proof. Again, we use the vertices of $\mathsf{G}(m+1,n)$ to represent the vertices of $\mathsf{MG}(m,n)$.

Case 1. If the parities of $m+1$ and n are identical, then we color each vertex $g(i,j)$ with the number $(i+j) \bmod 2$. This is a proper 2-coloring of $\mathsf{MG}(m,n)$. Hence $\mathsf{MG}(m,n)$ is a bipartite graph and has circular chromatic number 2.

Case 2.1. Assume that $m = 2p$ and $n = 2q$.

Since the length of the cycle $g(0,q)g(1,q)\cdots g(m,q)g(m{+}1,q)$ in $\mathsf{MG}(m,n)$ is $m+1$, its circular chromatic number is $2+\frac{1}{p}$. Hence $\chi^\circ(\mathsf{MG}(m,n)) \geq 2+\frac{2}{m}$. It is straightforward to check that the mapping $\alpha(g(i,q\pm j)) = (i+j)p \bmod (m+1)$, $0 \leq j \leq q$, is an $(m+1,p)$-coloring of $\mathsf{MG}(m,n)$, and hence $\chi^\circ(\mathsf{MG}(m,n)) = 2+\frac{2}{m}$.

Case 2.2. Assume that $m = 2p+1$ and $n = 2q+1$.

It is straightforward to check that the following mapping β is a $(2m+2,m)$-coloring of $\mathsf{MG}(m,n)$, and hence $\chi^\circ(\mathsf{MG}(m,n)) \leq 2+\frac{2}{m}$.

$$\beta(g(i,j)) = \begin{cases} im \bmod (2m+2) & \text{if } j \text{ is odd,} \\ (i+1)m+1 \bmod (2m+2) & \text{if } j \text{ is even.} \end{cases}$$

By Theorem 1, there exists an orientation D such that $\theta(\mathsf{MG}(m,n);D) = \chi^\circ(\mathsf{MG}(m,n)) - 1 \leq 1+\frac{2}{m} < 3$. Hence the flow ratio of the boundary of $\mathsf{MG}(m,n)$ is greater than 1 by Lemma 7. Since the boundary has length $2m+2$, its flow ratio must be at least $1+\frac{2}{m}$. This implies that $\chi^\circ(\mathsf{MG}(m,n)) = \theta(\mathsf{MG}(m,n);D)+1 \geq 2+\frac{2}{m}$. The theorem is thus proved. $\square$

References

[1] L. A. Goddyn, M. Tarsi, and C. Q. Zhang, On (k, d)-colorings and fractional nowhere-zero flows, J. Graph Theory 28(1998), 155 – 161.

[2] H. Hajiabolhassan and X. Zhu, Circular chromatic number of Kneser graphs, J. Combin. Theory Ser. B 88(2003), 299 – 303.

[3] A. Johnson, F. C. Holroyd, and S. Stahl, Multichromatic numbers, star chromatic numbers and Kneser graphs, J. Graph Theory 26(1997), 137 – 145.

[4] M. Kneser, Aufgabe 300, Jber. Deutsch. Math.-Verein. 58(1955), 27.

[5] Ko-Wei Lih and Daphne Der-Fen Liu, Circular chromatic numbers of some reduced Kneser graphs, J. Graph Theory 41(2002), 62 – 68.

[6] L. Lovász, Kneser's conjecture, chromatic number, and homotopy, J. Combin. Theory, Ser. A 25(1978), 319 – 324.

[7] A. Schrijver, Vertex-critical subgraphs of Kneser graphs, Nieuw Arch. Wisk. (3) 26(1978) 454 – 461.

[8] A. Vince, Star chromatic number, J. Graph Theory 12(1988), 551 – 559.

[9] X. Zhu, Star chromatic numbers and products of graphs, J. Graph Theory 16(1992), 557 – 569.

[10] X. Zhu, Circular chromatic number: a survey, Discrete Math. 299(2001), 371 – 410.