Erdős on Unit Distances and the Szemerédi–Trotter Theorems

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This is a concise review that attempts to show the vast influence of the work of Paul Erdős in a narrow area, the combinatorics of unit distances in geometry. This review tries to follow the history of the problems and cover the latest and strongest results, but cannot be complete. Excellent sources of further information are the book of Pach and Agarwal [47], and Chapter 17 in the Handbook of Combinatorics written by Erdős and Purdy [27]. Exercises in a book being written by Babai and Frankl [2] also cover many variants of the coloration problems that I discuss.

1 Coloring $\mathbb{R}^n$

E. Nelson and J. R. Isbell, and independently Erdős and H. Hadwiger, posed the following problem: *what is the minimum number of colors needed to color the points of $\mathbb{R}^n$ if points at unit distance apart must receive different colors?* This number is called the *chromatic number* of $\mathbb{R}^n$, and is denoted by $\chi(\mathbb{R}^n)$. This is exactly the chromatic number of the following graph, which is called the *unit distance graph*: $V = \mathbb{R}^n$ and join two vertices if their distance is one.

For $n = 2$, there has been no improvement on the estimates $4 \leq \chi(\mathbb{R}^2) \leq 7$ [36, 44]. We reproduce here the *Moser spindle*, a configuration of 7 points in the plane, with the property, that any 3 of the 7 points include 2 points at unit distance apart. The Moser spindle is evidence that the chromatic number of the plane is at least four.

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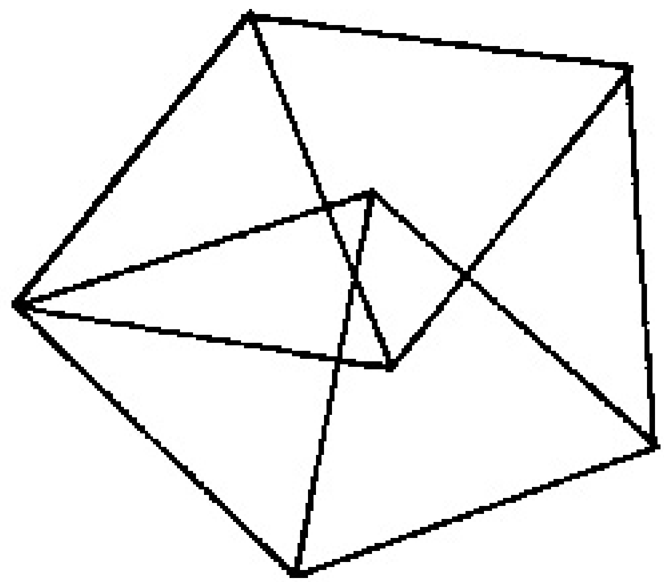

Fig. 1 : The Moser spindle. All eleven segments shown are of unit length.

I also show here two doubly periodic colorations of the plane. Points in a region are colored with the number assigned to the region, boundary points are colored with the color of any of their neighboring regions.

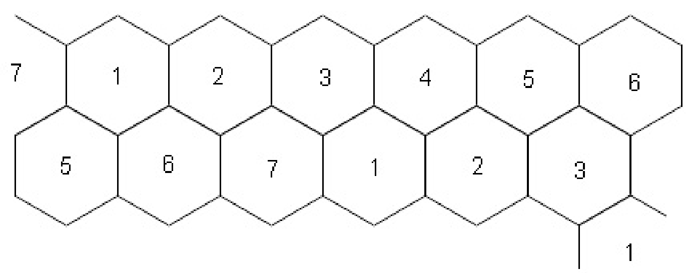

Fig. 2 : A good 7 coloration of the plane. The diameter of a hexagon is slightly less than 1.

There is no improvement on the lower bound $5 \leq \chi(\mathbb{R}^3)$ [39] either. However, D. Coulson [15] proved $\chi(\mathbb{R}^3) \leq 18$, and recently announced $\chi(\mathbb{R}^3) \leq 15$.

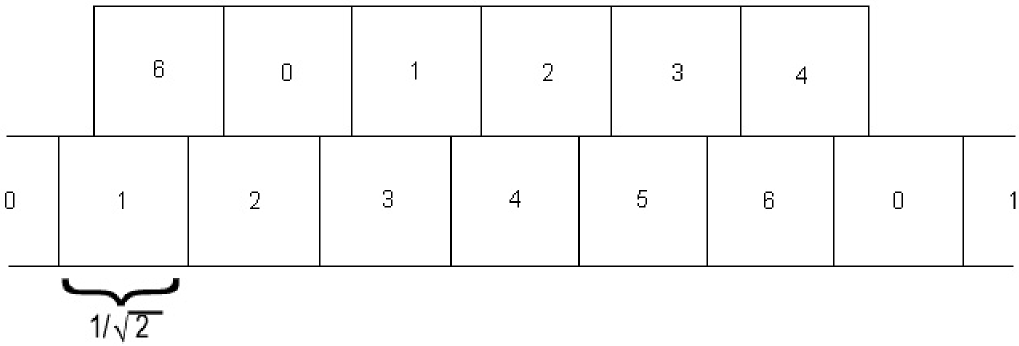

Fig. 3 : Another good 7 coloration of the plane. The diagonal of a square is 1.

It is not difficult to come up with an exponential upper bound for $\chi(\mathbb{R}^n)$. The best known upper bound, $(3+o(1))^n$, is due to Larman and Rogers [39]. They also showed that if there is a sphere packing in $\mathbb{R}^n$ of density $(2^{-.5}+o(1))^n$ as conjectured, then $\chi(\mathbb{R}^n) < (2^{1.5}+o(1))^n$.

2 Critical configurations in $\mathbb{R}^n$

To set a good lower bound for the chromatic number of a graph is a notoriously hard problem. D. G. Larman and C. A. Rogers [39] generalized the Moser spindle into a configurational principle. They looked at the inequality

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}, \tag{1}$$

i.e. the chromatic number of a graph G is at least the number of vertices divided by the independence number. This led them to the following definition: a set of M points makes an (M,D)-*critical configuration* in $\mathbb{R}^n$ if any $D+1$ of the M points contain some two points at unit distance apart. (Multiple points are allowed. Clearly any distance d can play the role of the unit distance. In this case we say that the configuration is critical for the distance d.) Obviously, the existence of an (M,D)-critical configuration in $\mathbb{R}^n$ implies $\chi(\mathbb{R}^n) \geq M/D$. All lower bounds for $\chi(\mathbb{R}^n)$ that I know of come in this way. However, Larman and Rogers [39] (based on earlier work of Hadwiger [35] and D. E. Raiskii [51]) proved a much stronger theorem:

If there exists an (M,D)-critical configuration in $\mathbb{R}^n$ $(n \geq 2)$, then in any partition of $\mathbb{R}^n$ into fewer than M/D classes, one class exhibits all positive real numbers as distances between some of its points.

For $n=2$, the best known (M,D)-critical configuration is the Moser spindle yielding $M/D = 3.5$. Although there is no improvement on the upper bound $\chi(\mathbb{R}^2) \leq 7$, it is remarkable that Stechkin [51], D. R. Woodall [62], and A. Soifer

[52, 53], found “better and better” partitions of the plane into six sets, such that none of the six sets exhibits all positive real numbers as distances between some of its points. In view of the cited theorem of Larman and Rogers, this means that there is no (M, D)-critical configuration in the plane with $M/D > 6$.

Larman and Rogers paid much attention to constructing good critical configurations in low dimensions, and many of the critical configurations that they discovered are still the best ones in the respective dimensions. I repeat here those constructions.

The *half-cube* is a $(16, 2)$-critical configuration with critical distance $\sqrt{2}$ in $\mathbb{R}^5$, its points are the 0-1 vectors of length 5 with even sum. The 56 vertices *7-dimensional Gosset polytope* can be obtained by putting signs in all the 8 ways to the characteristic vectors of the lines of the Fano plane. This is an $(56, 4)$-critical configuration with critical distance 2 in $\mathbb{R}^7$.

The *Moser–Raiskii spindle* is an $(n^2 + 2n - 1, n)$-critical configuration with critical distance 1 in $\mathbb{R}^n$. This is an n-dimensional generalization of the Moser spindle, its construction is the following. Take a regular simplex in $\mathbb{R}^n$ with side 1, specify one of its vertices, the *top* of the simplex, and reflect the point into the opposite face to obtain the *bottom* point. The resulting $(n + 2)$ vertices, including the bottom, make a *needle*. Take n needles whose bottoms coincide, and whose tops make a regular simplex of side 1. Take the common bottom with multiplicity $(n - 1)$.

The *half-cube spindle* and the *special Gosset spindle* can be defined by a further generalization of the spindle construction. Let us be given a $\mathcal{C}$ (M, D)-critical configuration in $\mathbb{R}^n$ with critical distance d, such that $\mathcal{C}$ is on a sphere of radius r. If $r \leq 2d/\sqrt{5}$, then $\mathcal{S}$, *the spindle of* $\mathcal{C}$, is an $(M^2 + 2DM - D^2, DM)$-critical configuration in $\mathbb{R}^{n+1}$ with critical distance d. The description is as follows. Add a top and a bottom to $\mathcal{C}$ in $\mathbb{R}^{n+1}$, such that both the bottom and the top are distance d apart from all vertices of $\mathcal{C}$, and the top and the bottom are mirror images of each other with respect to the center of the sphere containing $\mathcal{C}$. An $(M+2)$-vertex configuration is obtained, that we call a *needle*. Take now M needles whose bottoms coincide, and whose tops make a copy of $\mathcal{C}$. Take the tops with multiplicity D, the bottom in common with multiplicity $D(M - D)$, and all other vertices of the M needles with multiplicity one. This is the spindle $\mathcal{S}$.

If $r = 2d/\sqrt{5}$, then $\mathcal{T}$, *the special spindle of* $\mathcal{C}$, is an $(M + 2D, D)$-critical configuration with critical distance d in $\mathbb{R}^{n+1}$. Add a top and a bottom to $\mathcal{C}$, such that their distance from all vertices in $\mathcal{C}$ is d, and in addition, the distance of the top and the bottom is also d. Take the top and the bottom with multiplicity D, and all other points of $\mathcal{C}$ with multiplicity one to obtain the special spindle $\mathcal{T}$ of $\mathcal{C}$.

Now the *half-cube spindle* is the spindle of the 5-dimensional halfcube, and the *special Gosset spindle* is the special spindle of the 7-dimensional Gosset polytope.They give the best critical configurations that we know in $\mathbb{R}^6$ and $\mathbb{R}^8$.

Larman and Rogers in an appendix of their paper [39] cite an unpublisehed lemma of Erdős and Vera Sós. This lemma applied in a new context an observation of Zsigmond Nagy [46] and proved tight bounds on it. The configuration of Erdős and Vera Sós is the following. Consider in $\mathbb{R}^{n+1}$ all 0-1 vectors with exactly three 1's, so $M = \binom{n+1}{3}$. This configuration is on a hyperplane and therefore admits distance preserving embedding in $\mathbb{R}^n$. Regarding this as a critical configuration for distance 2, they showed

$$D = \begin{cases} n+1 & \text{if } n+1 \equiv 0 \bmod 4, \\ n & \text{if } n+1 \equiv 1 \bmod 4, \\ n-1 & \text{if } n+1 \equiv 2 \text{ or } 3 \bmod 4. \end{cases}$$

This configuration gave a quadratic lower bound for $\chi(\mathbb{R}^n)$. Erdős conjectured that $\chi(\mathbb{R}^n)$ grows exponentially in n. This conjecture was verified by P. Frankl and R. M. Wilson [31], who proved a strong intersection theorem for set systems, which actually generalizes the lemma of Erdős and Sós. Using this theorem, P. Frankl and R. M. Wilson exhibited in $\mathbb{R}^n$, in particular among its 0-1 vectors, an (M, D)-critical configuration with $M/D \geq (1.2 + o(1))^n$.

The Frankl–Wilson construction is the following: take any prime $p < n/2$ and consider in $\mathbb{R}^{n+1}$ all 0-1 vectors with exactly $2p-1$ 1's, so $M = \binom{n+1}{2p-1}$. Regarding this as a critical configuration for distance $\sqrt{2p}$, they showed $D \leq \binom{n+1}{p-1}$. This configuration is on a hyperplane again, and therefore admits distance preserving embedding in $\mathbb{R}^n$. Note that the Nagy–Erdős–T. Sós construction is a special case for $p = 2$, giving slightly improved bounds on D in this special case.

This Frankl–Wilson intersection theorem and construction is also the basis of two other breakthrough results. One is the best constructive lower bound for the Ramsey number $R(k, k)$ given in the same paper. The other is an unexpected counterexample to the Borsuk conjecture by J. Kahn and G. Kalai [38], who observed that the Borsuk problem can be "translated" by a surprising transformation into a critical distance problem. Recently A. M. Raigorodskii [49, 50] made a stronger counterexample to the Borsuk conjecture, which already works in 561 dimensions, and had an (M, D)-critical configuration in $\mathbb{R}^n$ with $M/D \geq (1.236 + o(1))^n$. This configuration, however, did not beat the old ones in low dimension.

3 The measurable chromatic number

One cannot apply (1) directly to the unit distance graph in $\mathbb{R}^n$. However, a substitute can be found if we restrict our interest to colorations of $\mathbb{R}^n$ with Lebesgue measurable color classes. Let the *measurable chromatic number* of

$\mathbb{R}^n$, $\chi^m(\mathbb{R}^n)$, denote the least number of colors of a good coloration using Lebesgue measurable color classes. Clearly $\chi(\mathbb{R}^n) \leq \chi^m(\mathbb{R}^n)$. Let us define a substitute for the independence number as

$$m_1(\mathbb{R}^n) = \sup_{X \subseteq B_n(r)} \limsup_{r \to \infty} \frac{\lambda(X \cap B_n(r))}{\lambda(B_n(r))}, \tag{2}$$

where λ is the n-dimensional Lebesgue measure, $B_n(r)$ is the ball of radius r around the origin in $\mathbb{R}^n$, and X is a Lebesgue measurable subset of $\mathbb{R}^n$, such that no two points of X are at unit distance apart. Then we have $\chi^m(\mathbb{R}^n) \geq 1/m_1(\mathbb{R}^n)$. A section was devoted to $m_1(\mathbb{R}^2)$ in the problem collection of W. Moser [45].

The study of $\chi^m(\mathbb{R}^n)$ started with K. J. Falconer's proof [28] of $\chi^m(\mathbb{R}^n) \geq n+3$. Note that in particular $\chi^m(\mathbb{R}^2) \geq 5$. It is obvious that $m_1(\mathbb{R}^n) \leq D/M$, if an (M, D)-critical configuration exists in $\mathbb{R}^n$. For $n = 2$, the best lower bound is $0.2293 \leq m_1(\mathbb{R}^2)$. The construction is just a slight modification of packing disks of radius $1/2$ into the vertices of a triangular lattice with edge length 2. Namely, the construction packs *tortoises* of diameter 1 into the vertices of a triangular lattice with edge length slightly less than 2. A tortoise is the union of a regular hexagon of diameter 1 and a disc centered at the midpoint of the hexagon with radius slightly less than .5. H. T. Croft [16] attributes this construction to L. Moser.

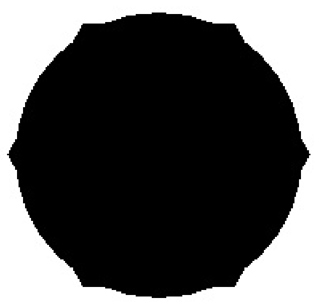

Fig. 4 : A tortoise.

Erdős conjectured that $m_1(\mathbb{R}^2) < .25$. If this conjecture is true, it gives a new proof for $\chi^m(\mathbb{R}^2) \geq 5$. It is easy to see that for any finite subgraph G of the unit distance graph of the plane on n vertices, $\alpha(G)/n \geq m_1(\mathbb{R}^2)$. Therefore Erdős also asked how small $\alpha(G)/n$ can be for such graphs.

For large n, I do not know of lower bounds to $m_1(\mathbb{R}^n)$ better than the reciprocal of the best upper bound for $\chi^m(\mathbb{R}^n)$ (which is always the best known upper bound for $\chi(\mathbb{R}^n)$ as well). I proved $m_1(\mathbb{R}^2) \leq 12/43$, $m_1(\mathbb{R}^3) \leq 7/37$, $m_1(\mathbb{R}^4) \leq 9/70$ in my thesis [58]. Later in joint work with Wormald [60] we introduced a configurational principle to estimate m_1. Using a computer search for configurations we estimated m_1 for $n \leq 24$. We obtained e.g. $\chi^m(\mathbb{R}^{24}) \geq 933$ (only $\chi(\mathbb{R}^{24}) \geq 178$ is known), and $\chi^m(\mathbb{R}^4) \geq 8$ (improving the Falconer bound). I give more details in the next Section. Even for $n = 2, 3, 4$, a tiny improvement was achieved on m_1. Larman and Rogers, and even earlier L. Moser [39], conjectured that $m_1(\mathbb{R}^n) \leq 2^{-n}$. This would imply $\chi^m(\mathbb{R}^n) \geq 2^n$.

The table below, following [39] and [60], summarizes the best lower bounds for the chromatic number and the measurable chromatic number in low dimensions. The fourth column gives the value of (M, D) for the best known configuration, while the fifth columns describes the configuration.

$n =$	$\chi(R^n) \geq$	$\chi^m(R^n) \geq$	(M, D)	Configuration
2	4	5	(7,2)	Moser spindle
3	5	6	(14,3)	Moser–Raiskii spindle
4	6	8	(23,4)	Moser–Raiskii spindle
5	8	11	(16,2)	Half-cube
6	10	15	(316,32)	Half-cube spindle
7	14	19	(56,4)	7-dimensional Gosset polytope
8	16	30	(64,4)	Special Gosset spindle
9	16	35	(64,4)	Special Gosset spindle
10	19	45	(165,9)	Erdős–T. Sós configuration
11	19	56	(220,12)	Erdős–T. Sós configuration
12	24	70	(286,12)	Erdős–T. Sós configuration
13	31	84	(364,12)	Erdős–T. Sós configuration
14	35	102	(455,13)	Erdős–T. Sós configuration
15	37	119	$(\binom{16}{5}, \binom{16}{2})$	Frankl–Wilson configuration
16	67	148	$(\binom{17}{5}, \binom{17}{2})$	Frankl–Wilson configuration
17	56	174	$(\binom{18}{5}, \binom{18}{2})$	Frankl–Wilson configuration
18	68	194	$(\binom{19}{5}, \binom{19}{2})$	Frankl–Wilson configuration
19	82	263	$(\binom{20}{5}, \binom{20}{2})$	Frankl–Wilson configuration
20	97	315	$(\binom{21}{5}, \binom{21}{2})$	Frankl–Wilson configuration
21	114	374	$(\binom{22}{5}, \binom{22}{2})$	Frankl–Wilson configuration
22	133	526	$(\binom{23}{5}, \binom{23}{2})$	Frankl–Wilson configuration
23	154	754	$(\binom{24}{5}, \binom{24}{2})$	Frankl–Wilson configuration
24	178	933	$(\binom{25}{5}, \binom{25}{2})$	Frankl–Wilson configuration

A theorem of Erdős and N. G. de Bruijn [9] asserts that if an infinite graph has finite chromatic number, then its chromatic number is the maximum chromatic number of its finite subgraphs. This result depends on the Axiom of Choice. Under this theorem, the value of $\chi(\mathbb{R}^n)$ is achieved by a finite subgraph of the unit distance graph. This is not necessarily true for $\chi^m(\mathbb{R}^n)$, although nobody has an example with $\chi(\mathbb{R}^n) \neq \chi^m(\mathbb{R}^n)$.

4 Problems related to the chromatic number of $\mathbb{R}^n$

The first type of generalization that I consider here allows more than one distances. Define the H-distance graph with $V = \mathbb{R}^n$ and join two vertices if their distance belongs to a set H. Denote the corresponding chromatic numbers by χ_H and χ_H^m, and the analogue of the independence number by m_H, if X is an independent set in the H-distance graph in (2).

Just for $n = 2$, Erdős asked about the behavior of $f(k) = \max_{|H|=k} \chi_H(\mathbb{R}^2)$. He knew that $f(k)/k > c\sqrt{\log k}$, but nothing better is known. The related question on the independence number is $\inf_{|H|=k} m_H(\mathbb{R}^2)$. For $k = 2$, I proved that $\inf_{|H|=2} m_H(\mathbb{R}^2) \le m_{\{1,\sqrt{3}\}}(\mathbb{R}^2) \le 2/11$, and also proved that if $a/b > 1.401$, then $m_{\{a,b\}}(\mathbb{R}^2) \le 1/4$ [57], by averaging an inclusion-exclusion formula over the group of rotations.

Combining some sieve optimizing arguments from [60] with my original paper [57], the general theorem is the following:

Let $P_1, P_2, ..., P_n$ be a configuration of n points in the plane. Let H denote a finite set of positive numbers, and $h = \min H$. Using the notations $p = |\{(i,j) : i < j, |\overline{P_iP_j}| < h/2, |\overline{P_iP_j}| \notin H\}|$ and $q = |\{(i,j) : i < j, |\overline{P_iP_j}| \ge h/2, |\overline{P_iP_j}| \notin H\}|$, one has $m_H(\mathbb{R}^2) \le \frac{\binom{t+1}{2}}{tn-p-q/2}$, where $t = \lceil \frac{2p+q}{n} \rceil$.

Now the the upper bound $m_{\{a,b\}} < (\mathbb{R}^2) \le 1/4$ claimed above follows from the existence of the following configurations:

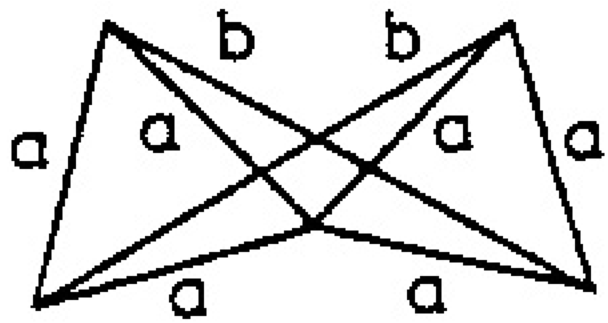

Fig. 5 : Configuration providing upper bound for $m_{\{a,b\}}$ if $\frac{1}{2}\sqrt{5+2\sqrt{3}} \ge \frac{b}{a} \ge \frac{1}{4}\left(\sqrt{13}+\sqrt{15}\right)$.

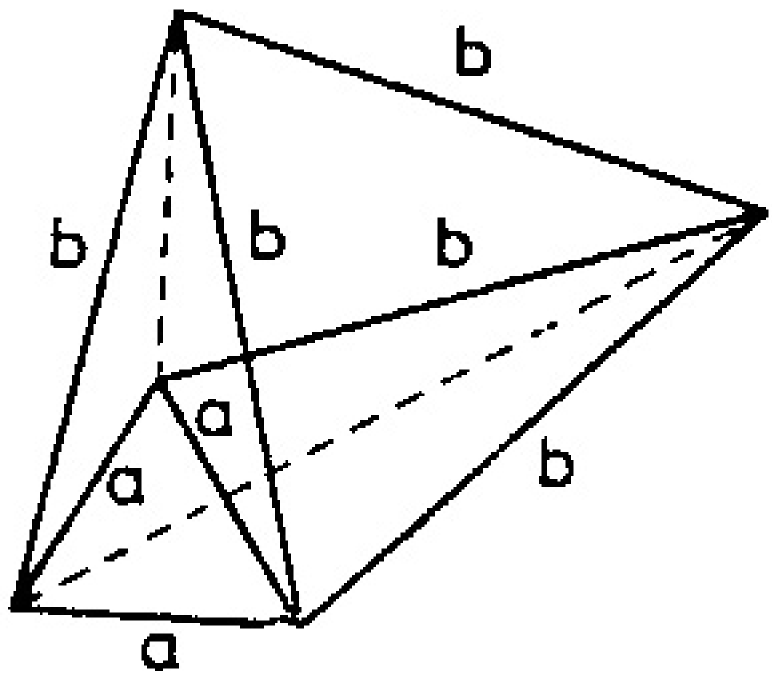

Fig. 6 : Configuration providing upper bound for $m_{\{a,b\}}$ if $\frac{b}{a} \geq \frac{1}{2}\sqrt{5+2\sqrt{3}}$.

Unfortunately, I was never able to find a configuration bounding $m_{\{a,b,c\}}$ for independent a, b, c in a non-trivial way using this configurational principle.

These problems turn even harder if H is infinite. I conjectured in my very first paper [56] that if H is not bounded from above, then $m_H(\mathbb{R}^2) = 0$. Erdős liked this conjecture and made it known. This conjecture was proved first by Y. Katznelson and B. Weiss [33], then by K. J. Falconer and J. M. Marstrand [30], and subsequently a generalization was proved by J. Bourgain [7].

A well-known theorem of Steinhaus asserts that if X is a set of positive Lebesgue measure in $\mathbb{R}^n$, then there is an $\epsilon > 0$ such that any translate of the set X to a distance at most ϵ intersects the set X. Therefore, if $\inf H = 0$, then $m_H(\mathbb{R}^2) = 0$. I made the following "continuity conjecture": if $\inf H = 0$, then $\lim_{\delta \to 0} m_{H \cap (\delta, \infty)}(\mathbb{R}^2) = 0$. This conjecture was proved by Falconer [29].

A second type of analogue is the following. Change the vertex set from $\mathbb{R}^n$ to a subset of $\mathbb{R}^n$. In particular, much attention have been paid to the sphere S^{n-1} of radius 1 in $\mathbb{R}^n$.

Consider $H = [\alpha, 2]$ for $0 < \alpha < 2$ and the distance graph that H defines on S^{n-1}. This graph, the *Borsuk graph*, was introduced and studied by Erdős and A. Hajnal [25]. It follows from Borsuk's Theorem [6] (and in fact, is equivalent to it), that $\chi_H(S^{n-1}) \geq n+1$. Combining this with the Erdős–de Bruijn Theorem [9] on the chromatic number of infinite graphs, one obtains one of the earliest constructions for finite graphs with high chromatic number and high odd girth.

Erdős and R. L. Graham conjectured, that the chromatic number is already large if one uses only the single distance α instead of the whole H. This was proved by Lovász [42], who showed that for any $\sqrt{2(n+1)/n} < \alpha < 2$ (i.e. when α exceeds the side length of the regular simplex inscribed the sphere), $n \leq \chi_\alpha(S^{n-1}) \leq n+1$, and for every $n \geq 2$ showed the existence of infinitely many α's with $\chi_\alpha(S^{n-1}) = n+1$. Lovász *did not use critical configurations*, he used his topological framework providing lower bounds for the chromatic numbers of graphs. This framework was actually developed in order to set a tight lower bound for the chromatic number of the *Kneser graph*, which is the "discrete analogue" of the Borsuk graph [41]. For a smaller α, Frankl and Wilson gave exponential lower bound for $\chi_\alpha(S^{n-1})$ [31].

The measurable analogue of the independence number generalizes for spheres as well. Let $m_H^{(n-1)}(r)$ denote the supremum of the independence ratios

$$m_H^{(n-1)}(r) = \sup_{X \subseteq S^{n-1}(r)} \frac{\lambda(X)}{\lambda(S^{n-1}(r))}, \tag{3}$$

where λ is the $(n-1)$-dimensional Lebesgue measure, $S^{n-1}(r)$ is the sphere of radius r around the origin in $\mathbb{R}^n$, and X is a Lebesgue measurable subset of $S^{n-1}(r)$ such that no two points of X have a distance belonging to H. N. C. Wormald and I [60] discovered a recursive formula to give upper bounds on $m_H^{(n)}(r)$ and used this technique for $r = \infty$ to give upper bounds to $m_1(\mathbb{R}^n)$ and lower bounds to $\chi^m(\mathbb{R}^n)$. Our configurational principle is the following:

Let $P_1, P_2, ..., P_m$ denote a point configuration on $S^{n-1}(r)$. Define

$$\Sigma = \sum_{\substack{i<j \\ |\overline{P_iP_j}| \notin H}} m_H^{(n-2)}\left(|\overline{P_iP_j}|\sqrt{1-(|\overline{P_iP_j}|/2r)^2}\right),$$

with $m_H^{(n-2)}(0) = 1$. Then, with $t = \lceil \frac{2\Sigma}{M} \rceil$,

$$m_H^{(n-1)}(r) \leq \frac{\binom{t+1}{2}}{tM - \Sigma}.$$

Another well-studied problem is, how heavily the chromatic number of the plane depends on subgraphs of the unit distance graph having small girth. Erdős asked if there is any triangle-free subgraph of the unit distance graph in $\mathbb{R}^2$ which is not 3-colorable. N. C. Wormald [63] showed the existence of such a graph on 6448 vertices, later P. O'Donnell [17] on 56, finally K. B. Chilakamarri [11] on 47 vertices. Recently O'Donnell [18] announced the construction of arbitrary large girth subgraphs of the unit distance graph in $\mathbb{R}^2$ which are not 3-colorable.

5 The problem of distinct distances and unit distances

Erdős says [23] "My most striking contribution to geometry is, no doubt, my problem on the number of distinct distances". This was the following pair of problems, posed in 1946 [20] : what is $f(n)$, the maximum number of unit distances among n points in the plane; and what is $g(n)$, the minimum number of distinct distances among n points in the plane? Erdős offered USD 500 for the magnitude of any of these functions. It is clear that

$$f(n)g(n) \geq \binom{n}{2}. \tag{4}$$

Erdős showed [20] that $n^{1+c/\log\log n} < f(n)$ and conjectured that this was about the true magnitude of $f(n)$. He proved that $f(n)$ is at most $cn^{3/2}$. S. Józsa and E. Szemerédi [37] improved this bound to $o(n^{3/2})$, J. Beck and J. Spencer [5] further improved the bound to $n^{1.44\cdots}$. Finally, Spencer, Szemerédi, and Trotter [54] achieved the best known bound, $cn^{4/3}$.

Erdős showed [20] that $g(n) < cn/\sqrt{\log n}$ by considering the lattice points in a large disk, and conjectured that this was about the true magnitude of $g(n)$. Erdős showed the lower bound $g(n) > \sqrt{n}$, L. Moser [43] showed $n^{2/3}$, F. R. K. Chung [12] showed $n^{5/7}$, Beck [4] showed $n^{58/81-\epsilon}$, finally Chung, Szemerédi and Trotter [13] showed $n^{4/5}/(\log n)^c$. As they remarked, they did not prove the existence of a single point from which so many distinct distances start. For this modified problem the best known result was $cn^{3/4}$ by K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir and E. Welzl [14]. The problem of distinct distances has already been studied in Minkowski spaces [55].

The following result on crossing numbers of graphs was conjectured by Erdős and R. K. Guy [24, 34], and later independently proved by F. T. Leighton [40], and M. Ajtai, V. Chvátal, M. Newborn and E. Szemerédi [1]:

Any simple graph on n vertices and $m \geq 4n$ edges requires at least $.01m^3/n^2$ crossings if drawn on the plane.

I gave simple and transparent proofs [59] for $f(n) < cn^{4/3}$ and $g(n) > cn^{4/5}$ using this theorem. For the latter estimation, I find $cn^{4/5}$ distinct distances starting from a particular point. Unfortunately, my method cannot give a better upper bound on $f(n)$, since Erdős, D. Hickerson and J. Pach [26] gave $cn^{4/3}$ unit distances among n points on a sphere of radius $1/\sqrt{2}$. Any direct application of my method to the plane would work on the sphere as well.

My method also gave a simple and transparent proof to the following Szemerédi–Trotter theorem [61, 3] on incidences:

For n points and l lines in the Euclidean plane, such that $n < l^2$ and $l < n^2$, *the number of incidences among the points and lines is at most* $c(nl)^{2/3}$.

This theorem, originally conjectured by Erdős and Purdy [22], is a fundamental result in discrete and combinatorial geometry. It is known to be tight in its whole range [3, 47]. Other proofs for this theorem are in [14, 32]. An important, frequently used corollary of the theorem is the following:

Given n *points and* l *lines in the Euclidean plane, such that for some* $2 \leq k \leq \sqrt{n}$, *every line is incident to at least* k *of the points. Then* $l \leq cn^2/k^3$.

It is worth noting that J. Beck, who proved a somewhat weaker result than this, used the weaker result to resolve a number of open problems in combinatorial geometry [3].

D. de Caen and I [10] proved the following stronger theorem: the number of 3-paths in the incidence bipartite graph of such n points and l lines is at most cnl. This theorem implies the Szemerédi–Trotter theorem on incidences through Atkinson's inequality. D. de Caen and I made an even stronger conjecture, that the number of 6-cycles in the incidence bipartite graph is at most nl. E. Moorehouse offered USD 50 for a counterexample to this conjecture.

It is a curious fact why the Szemerédi–Trotter theorem on incidences does hold. For an affine geometry over a finite field the theorem badly fails if we count incidences among all lines and all points. For which fields does the Szemerédi–Trotter theorem on incidences hold? In particular, Elekes asked if it holds in the complex geometry. Recently, G. Elekes and Csaba Tóth [19] gave a positive answer to this question. Their proof gave step-by-step a complex analogue of the original proof of Szemerédi and Trotter [61].

Functions $f(n)$ and $g(n)$ can be investigated in dimension $d > 2$ as well. For $d = 3$ the best bounds are $cn^{4/3}\log\log n < f_3(n)$ by Erdős [21] and $f_3(n) < n^{3/2+o(1)}$ by Clarkson *et al.* [14]. It follows from (4) that $g_3(n) \geq n^{1/2-o(1)}$. According to the Lenz construction, $f_d(n) > cn^2$ for $d \geq 4$. Recently Braß [8] determined $f_4(n)$ exactly in dimension $d = 4$.

For estimating $g_d(n)$ in higher dimension d, Erdős [20] had the upper bound: $g_d(n) \leq cn^{2/d}$. The best current lower bound is $cn^{\frac{1}{d-1}-o(1)} \leq g_d(n)$, as d is fixed and $n \to \infty$. This lower bound is sketched in [47] p. 197, using $g_3(n) \geq n^{1/2-o(1)}$ from [14] as base case for an inductive proof.

The Szemerédi–Trotter theorem has variants for other curves than straight lines, in this case the *degree of freedom* of the curve family determines the exponent, see J. Pach and M. Sharir [48].

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