SYZYGYIES SPRING 2018 A. KUSTIN CLASS NOTES

1. REGULAR SEQUENCES, THE KOSZUL COMPLEX, "WHAT MAKES A COMPLEX EXACT?"

Let R be a commutative Noetherian ring and M be a finitely generated R-module. We want to learn the resolution of M by free R-modules. Typically, R is local and we want the minimal resolution of M; or R is graded over a field $\boldsymbol{k}$, M is graded, and we want the minimal homogeneous resolution of M.

In any case, the resolution looks like a complex

$$\mathbb{F}: \quad \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0,$$

with each F_i a finitely generated free R-module, d_i an R-module homomorphism, and

$$\mathrm{H}_i(\mathbb{F}) \cong \begin{cases} M, & \text{if } i=0, \text{ and} \\ 0, & \text{if } 1 \le i, \end{cases}$$

where $\mathrm{H}_i(\mathbb{F}) = \ker d_i / \operatorname{im} d_{i+1}$.

If $(R, \mathfrak{m})$ is local, then $\mathbb{F}$ is minimal if $\operatorname{im} d_i \subseteq \mathfrak{m}F_{i-1}$ for all i.

If R is graded, then $\mathbb{F}$ is homogeneous if

- $d_i(\theta)$ is homogeneous for each homogeneous element θ in F_i, and
- $\deg(d_i(\theta)) = \deg(\theta)$ for each homogeneous element θ in F_i.

The homogeneous resolution $\mathbb{F}$ is minimal if $d_i(F_i) \subseteq \mathfrak{m}F_{i-1}$, where $\mathfrak{m}$ is the maximal homogeneous ideal of R.

1.A. Regular sequences.

The easiest ideals to resolve are ideals generated by regular sequences.

Definition 1.1. Let R be a commutative Noetherian ring, M be an R-module, and $f_1, \dots, f_n$ be elements of R. The sequence $f_1, \dots, f_n$ is a regular sequence on M if

- $(f_1, \dots f_n)M \subsetneq M$,
- f_1 is regular on M,
- f_2 is regular on $M/(f_1)M$,
- $\vdots$
- f_n is regular on $M/(f_1, \dots, f_{n-1})M$.

Example 1.2. The elements $x_1, \dots, x_n$ are a regular sequence on $R = \boldsymbol{k}[x_1, \dots, x_n]$.

Definition 1.3. If $f_1, \dots, f_n$ are elements of the ring R, then the Koszul complex on $f_1, \dots, f_n$ is the complex

$$0 \to \bigwedge^n F \xrightarrow{d_n} \bigwedge^{n-1} F \xrightarrow{d_{n-1}} \bigwedge^{n-2} F \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_2} \bigwedge^1 F \xrightarrow{d_1} \bigwedge^0 F \to 0,$$

where $F = \bigoplus_{i=1}^n Re_i$ and

$$d_j(e_{i_1} \wedge \ldots \wedge e_{i_j}) = \sum_{k=1}^{j} (-1)^{k+1} f_{i_k} e_{i_1} \wedge \ldots \wedge \widehat{e_{i_k}} \wedge \ldots \wedge e_{i_j}.$$

Theorem 1.4. *If $f_1, \ldots, f_n$ is a regular sequence in the ring R, then the Koszul complex on $f_1, \ldots, f_n$ is a resolution of $R/(f_1, \ldots, f_n)$. This resolution is minimal if $(R, \mathfrak{m})$ is local and $f_1, \ldots, f_n$ are in $\mathfrak{m}$. This resolution is homogeneous and minimal if R is graded and the f's are homogeneous of positive degree.*

Examples 1.5. If f_1, f_2, f_3 is a regular sequence in R on R, then

$$0 \to R \xrightarrow{f_1} R \to 0$$

is a resolution of $R/(f_1)$;

$$0 \to R \xrightarrow{\begin{bmatrix} -f_2 \\ f_1 \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} f_1 & f_2 \end{bmatrix}} R \to 0$$

is a resolution of $R/(f_1, f_2)$; and

$$0 \to Re_1 \wedge e_2 \wedge e_3 \xrightarrow{\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}} \begin{matrix} Re_2 \wedge e_3 \\ \oplus \\ Re_3 \wedge e_1 \\ \oplus \\ Re_1 \wedge e_2 \end{matrix} \xrightarrow{\begin{bmatrix} 0 & f_3 & -f_2 \\ -f_3 & 0 & f_1 \\ f_2 & -f_1 & 0 \end{bmatrix}} \begin{matrix} Re_1 \\ \oplus \\ Re_2 \\ \oplus \\ Re_3 \end{matrix} \xrightarrow{\begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}} R \to 0$$

is a resolution of $R/(f_1, f_2, f_3)$.

Outline of a proof of Theorem 1.4. The proof is by induction. We have already seen that the assertion holds for small values of n. Suppose the assertion holds for $n-1$. Write $F = F' \oplus Re_n$, where $F' = \bigoplus_{i=1}^{n-1} Re_i$. Observe that

$$\textstyle\bigwedge^i F = \bigwedge^i (F' \oplus Re_n) = (\bigwedge^i F') \oplus (\bigwedge^{i-1} F' \otimes Re_n)$$

and that the commutative diagram

(1.5.1)
$$\begin{tikzcd}
 & & & & & 0 \arrow[d] & \\
T: & \cdots \arrow[r, "t_3"] & F'_2 \arrow[r, "t_2"] \arrow[d, "f_n"] & F'_1 \arrow[r, "t_1"] \arrow[d, "f_n"] & F'_0 \arrow[r] \arrow[d, "f_n"] & R/(f_1,\dots,f_{n-1}) \arrow[r] \arrow[d, "f_n"] & 0 \\
B: & \cdots \arrow[r, "b_3"] & F'_2 \arrow[r, "b_2"] & F'_1 \arrow[r, "b_1"] & F'_0 \arrow[r] & R/(f_1,\dots,f_{n-1}) \arrow[r] \arrow[d] & 0 \\
 & & & & & R/(f_1,\dots,f_n) \arrow[d] & \\
 & & & & & 0 &
\end{tikzcd}$$

has exact rows and the right column is exact.

One can calculate by hand that the mapping cone of (1.5.1) is exact and that this mapping cone is isomorphic to the Koszul complex on $f_1,\dots,f_n$ **OR** one can use the long exact sequence of homology that corresponds to a short exact sequence of complexes. (One gets a little more information from the long exact sequence; but we do not need that information here.)

At any rate, the mapping cone of the map of complexes

$$\begin{tikzcd}
T: & \cdots \arrow[r, "t_3"] & T_2 \arrow[r, "t_2"] \arrow[d, "c_2"] & T_1 \arrow[r, "t_1"] \arrow[d, "c_1"] & T_0 \arrow[r] \arrow[d, "c_0"] & 0 \\
B: & \cdots \arrow[r, "b_3"] & B_2 \arrow[r, "b_2"] & B_1 \arrow[r, "b_1"] & B_0 \arrow[r] & 0
\end{tikzcd}$$

is

$$M: \quad \cdots \xrightarrow{m_3=\begin{bmatrix} t_2 & 0 \\ c_2 & -b_3 \end{bmatrix}} \underbrace{\begin{matrix} T_1 \\ \oplus \\ B_2 \end{matrix}}_{M_2} \xrightarrow{m_2=\begin{bmatrix} t_1 & 0 \\ c_1 & -b_2 \end{bmatrix}} \underbrace{\begin{matrix} T_0 \\ \oplus \\ B_1 \end{matrix}}_{M_1} \xrightarrow{m_1=\begin{bmatrix} c_0 & -b_1 \end{bmatrix}} \underbrace{B_0}_{M_0}.$$

The long exact sequence of homology that corresponds to a mapping cone is

$$\cdots \to \mathrm{H}_1(T) \to \mathrm{H}_1(B) \to \mathrm{H}_1(M) \to \mathrm{H}_0(T) \to \mathrm{H}_0(B) \to \mathrm{H}_0(M) \to 0.$$

One can prove that this long sequence is exact by hand, or by using the long exact sequence of homology that corresponds to the short exact sequence of complexes:

$$\begin{tikzcd}
0 \arrow[d] & & & & \\
B \arrow[d] & \cdots \arrow[r, "-b_3"] & B_2 \arrow[r, "-b_2"] \arrow[d, "\begin{bmatrix}0\\1\end{bmatrix}"] & B_1 \arrow[r, "-b_1"] \arrow[d, "\begin{bmatrix}0\\1\end{bmatrix}"] & B_0 \arrow[d, "\begin{bmatrix}0\\1\end{bmatrix}"] \\
M \arrow[d] & \cdots \arrow[r, "m_3=\begin{bmatrix}t_2 & 0\\ c_2 & -b_3\end{bmatrix}"] & \begin{matrix}T_1\\ \oplus\\ B_2\end{matrix} \arrow[r, "m_2=\begin{bmatrix}t_1 & 0\\ c_1 & -b_2\end{bmatrix}"] \arrow[d, "\begin{bmatrix}1 & 0\end{bmatrix}"] & \begin{matrix}T_0\\ \oplus\\ B_1\end{matrix} \arrow[r, "m_1=\begin{bmatrix}0 & 0\\ c_0 & -b_1\end{bmatrix}"] \arrow[d, "\begin{bmatrix}1 & 0\end{bmatrix}"] & \begin{matrix}T_{-1}\\ \oplus\\ B_0\end{matrix} \arrow[d, "\begin{bmatrix}1 & 0\end{bmatrix}"] \\
T[-1] \arrow[d] & \cdots \arrow[r, "t_2"] & T_1 \arrow[r, "t_1"] & T_0 \arrow[r, "0"] & T_{-1} \\
0. & & & &
\end{tikzcd}$$

□

1.B. **Use** Hom **(and** Ext**) to detect regular sequences.** The next goal is "What makes a complex exact?", which is a theorem by Buchsbaum and Eisenbud. The theorem says: the complex $\mathbb{F}$ is exact if and only if

- a linear algebra condition and
- a condition about regular sequences.

We need to learn a little more about each of these topics before we are able to appreciate this theorem.

We start with the regular sequences.

Observation 1.6. *Let M be a non-zero finitely generated module over the Noetherian ring R, and let I be an ideal in R. Then $\operatorname{Hom}_R(R/I, M) = 0$ if and only if there is an element $x \in I$ with x regular on M.*

Proof. ($\Leftarrow$) Assume x is an element of I with x regular on M. Prove $\operatorname{Hom}_R(R/I, M) = 0$.

Apply $\operatorname{Hom}_R(R/I, -)$ to the exact sequence

$$0 \to M \xrightarrow{x} M \to M/(x)M \to 0$$

to get the exact sequence

$$0 \to \operatorname{Hom}_R(R/I, M) \underbrace{\xrightarrow{x}}_{0} \operatorname{Hom}_R(R/I, M) \to \operatorname{Hom}_R(R/I, M/(x)M).$$

Conclude $\operatorname{Hom}_R(R/I, M) = 0$.

($\Rightarrow$) Assume I is contained in the zero divisors on M. Prove $\operatorname{Hom}_R(R/I, M) \neq 0$.

• We will use some facts from commutative algebra. If M is a non-zero finitely generated module over a Notherian ring, then there is a finite set of prime ideals of R (called the set of associated primes of M, denoted $\operatorname{Ass} M$) such that the following statements hold.

Fact 1.7. *The set of zero divisors on M is equal to $\cup_{P \in \operatorname{Ass} M} P$.*

Fact 1.8. *If P is a prime ideal of R, then P is an associated prime of M if and only if $P = \operatorname{ann}_R(m)$ for some element m of M.*

Fact 1.9. *Every prime ideal in R which is minimal over the annihilator of M is an associated prime of M.*

Fact 1.10. *If I is an ideal of R and every element of I is a zero divisor on M, then I is contained in an associated prime of M.*

Facts 1.7, 1.8, and 1.9 are about “Primary Decomposition”; this is the work of Emmy Noether. Fact 1.10 is usually called the “Prime Avoidance Lemma”; the Prime Avoidance Lemma is true more generally than recorded above.

Return to the proof. The ideal I is contained in the set of zero divisors on M; hence there is an associated prime ideal P of M with $I \subseteq P$; this associated prime is equal to $\operatorname{ann}(m)$ for some m in M. Observe that $1 \mapsto m$ is a non-zero element of $\operatorname{Hom}_R(R/I, M)$. $\square$

Definition 1.11. Let R be a ring, I be an ideal in R, and M be an R-module with $IM \neq M$. The grade in I on M (denoted $\operatorname{grade}(I, M)$) is the length of the longest regular sequence in I on M. We write $\operatorname{grade}(I)$ to mean $\operatorname{grade}(I, R)$. If $(R, \mathfrak{m})$ is Noetherian and local and M is a non-zero finitely generated R module, then $\operatorname{grade}(\mathfrak{m}, M)$ is also denoted $\operatorname{depth} M$.

Remark 1.12. If R is Noetherian, M is finitely generated, and $IM \neq M$, then $\operatorname{grade}(I, M)$ is finite. Indeed, if $x_1, x_2, \dots$ is a regular sequence in I on M, then

$$(x_1)M \subsetneq (x_1, x_2)M \subsetneq \cdots$$

If equality occurred at spot i, then $x_i M$ would be contained in $(x_1, \dots, x_{i-1})M$ with x_i regular on $M/(x_1, \dots, x_{i-1})M$. This would force $M \subseteq (x_1, \dots, x_{i-1})M$ which has been ruled out.

Theorem 1.13. *Let M be a finitely generated module over the Noetherian ring R, and let I be an ideal in R with $IM \neq M$. The following statements hold.*

(a) $\operatorname{grade}(I, M) = \min\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}$,

(b) *every maximal regular sequence in I on M has the same length, and*

(c) $\operatorname{grade}(I, M) \leq \operatorname{pd}_R R/I$.

Lemma 1.14. *Let M be a finitely generated module over the Noetherian ring R, and let I be an ideal in R. If $x_1, \dots x_n$ is a regular sequence on M in I, then*

$$\operatorname{Ext}^i_R(R/I, M) \cong \begin{cases} 0, & \text{if } 0 \leq i \leq n-1, \text{ and} \\ \operatorname{Hom}_R(R/I, M/(x_1, \dots, x_n)M), & \text{if } i = n. \end{cases}$$

Proof. The proof is by induction on n. We start with $n = 1$. Apply $\operatorname{Hom}_R(R/I, -)$ to the short exact sequence of R-modules

$$0 \to M \xrightarrow{x_1} M \to M/x_1M \to 0$$

to obtain the long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_R(R/I, M) &\underbrace{\xrightarrow{x_1}}_{0} \operatorname{Hom}_R(R/I, M) \to \operatorname{Hom}_R(R/I, M/x_1M) \\ \to \operatorname{Ext}^1_R(R/I, M) &\underbrace{\xrightarrow{x_1}}_{0} \operatorname{Ext}^1_R(R/I, M) \to \operatorname{Ext}^1_R(R/I, M/x_1M) \cdots . \end{aligned}$$

Conclude that $\operatorname{Hom}_R(R/I, M) = 0$ and

$$0 \to \operatorname{Ext}^i_R(R/I, M) \to \operatorname{Ext}^i_R(R/I, M/x_1M) \to \operatorname{Ext}^{i+1}_R(R/I, M) \to 0 \tag{1.14.1}$$

is exact for $0 \leq i$. In particular, when $i = 0$ in (1.14.1) one obtains

$$\operatorname{Hom}_R(R/I, M/x_1M) \cong \operatorname{Ext}^1_R(R/I, M).$$

We have established the case $n = 1$.

Suppose, by induction, that the assertion holds for $n-1$. Apply the induction hypothesis to the regular sequence $x_2 \dots, x_n$ on the module M/x_1M to conclude that

$$\operatorname{Ext}^i_R(R/I, M/x_1M) \cong \begin{cases} 0, & \text{if } 0 \leq i \leq n-2, \text{ and} \\ \operatorname{Hom}_R(R/I, \underbrace{(M/x_1)/(x_2, \dots, x_n)(M/x_1)}_{M/(x_1,\dots,x_n)M}), & \text{if } i = n-1. \end{cases}$$

Plug $\operatorname{Ext}^i_R(R/I, M/x_1M) = 0$ for $0 \leq i \leq n-2$ into (1.14.1) to see $\operatorname{Ext}^i_R(R/I, M) = 0$ for $0 \leq i \leq n-1$. At $i = n-1$, (1.14.1) gives

$$0 \to \underbrace{\operatorname{Ext}^{n-1}_R(R/I, M)}_{0} \to \underbrace{\operatorname{Ext}^{n-1}_R(R/I, M/x_1M)}_{\operatorname{Hom}_R(R/I, M/(x_1,\dots,x_n)M)} \to \operatorname{Ext}^n_R(R/I, M) \to 0,$$

and this concludes the proof of the Lemma. □

The proof of Theorem 1.13. Let $x_1, \dots x_n$ be a maximal regular sequence in I on M. We have shown that

$$\operatorname{Ext}^i(R/I, M) = \begin{cases} 0, & \text{if } 0 \leq i \leq n-1 \\ \operatorname{Hom}_R(R/I, M/(x_1, \dots, x_n)M), & \text{if } i = 0. \end{cases}$$

The regular sequence is maximal; so, Observation 1.6 yields that

$$\operatorname{Hom}_R(R/I, M/(x_1, \dots, x_n)M) \neq 0.$$

This completes the proof of (a) and (b).

(c) Let $\mathbb{F}$ be a projective resolution of R/I of length $\operatorname{pd}_R R/I$. Use

$$\operatorname{Ext}^i_R(R/I, M) = \operatorname{H}^i(\operatorname{Hom}(\mathbb{F}, M))$$

is to compute

$$\operatorname{Ext}^i_R(R/I, M) = 0 \quad \text{for } \operatorname{pd}_R R/I + 1 \leq i.$$

We know $\operatorname{grade}(I, M)$ is finite and

$$\operatorname{Ext}_R^{\operatorname{grade}(I,M)}(R/I, M) \neq 0.$$

Conclude $\operatorname{grade}(I, M) \leq \operatorname{pd}_R R/I$.

Doodle 1.15. Let R be a Noetherian ring, $I \subseteq R$ be an ideal, and

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of finitely generated R-modules. The following statements hold.

(a) $\min\{\operatorname{grade}(I, A), \operatorname{grade}(I, C)\} \leq \operatorname{grade}(I, B)$
(b) $\min\{\operatorname{grade}(I, B), \operatorname{grade}(I, C) + 1\} \leq \operatorname{grade}(I, A)$
(c) $\min\{\operatorname{grade}(I, A) - 1, \operatorname{grade}(I, B)\} \leq \operatorname{grade}(I, C)$

Proof. (a) If $i < \min\{\operatorname{grade}(I, A), \operatorname{grade}(I, C)\}$, then the exact sequence

$$\operatorname{Ext}_R^i(R/I, A) \to \operatorname{Ext}_R^i(R/I, B) \to \operatorname{Ext}_R^i(R/I, C)$$

yields $\operatorname{Ext}_R^i(R/I, B) = 0$; hence,

$$\min\{\operatorname{grade}(I, A), \operatorname{grade}(I, C)\} \leq \operatorname{grade}(I, B).$$

(b) If $i < \min\{\operatorname{grade}(I, B), \operatorname{grade}(I, C) + 1\}$, then

$$i < \operatorname{grade}(I, B) \quad \text{and} \quad i - 1 < \operatorname{grade}(I, C);$$

hence the exact sequence

$$\operatorname{Ext}_R^{i-1}(R/I, C) \to \operatorname{Ext}_R^i(R/I, A) \to \operatorname{Ext}_R^i(R/I, B)$$

yields $\operatorname{Ext}_R^i(R/I, A) = 0$ and

$$\min\{\operatorname{grade}(I, B), \operatorname{grade}(I, C) + 1\} \leq \operatorname{grade}(I, A).$$

(c) If $i < \min\{\operatorname{grade}(I, A) - 1, \operatorname{grade}(I, B)\}$, then

$$i + 1 < \operatorname{grade}(I, A) \quad \text{and} \quad i < \operatorname{grade}(I, B)$$

and the exact sequence

$$\operatorname{Ext}_R^i(R/I, B) \to \operatorname{Ext}_R^i(R/I, C) \to \operatorname{Ext}_R^{i+1}(R/I, A)$$

yields $\operatorname{Ext}_R^i(R/I, C) = 0$ and

$$\min\{\operatorname{grade}(I, A) - 1, \operatorname{grade}(I, B)\} \leq \operatorname{grade}(I, C).$$

□

1.C. **The linear algebra aspect of resolutions.**

Definition 1.16. Let R be a commutative Noetherian ring and $\phi : F \to G$ be an R-module homomorphism between finitely generated free R-modules.

(a) For each integer i, let $I_i(\phi)$ be the ideal generated by the $i \times i$ minors of (some matrix representation of) ϕ. (Of course, $I_i(\phi)$ is also equal to the image of the map

$$\bigwedge^i F \otimes_R \bigwedge^i G^* \to R$$

which is induced by ϕ; hence, $I_i(\phi)$ is independent of the choice of bases.)

(b) The rank of ϕ is the largest index i with $I_i(\phi) \neq 0$.

Folklore 1.17. Let R be a commutative Noetherian ring and M be a finitely generated R-module. The following statements about M are equivalent; and, if they hold, then M is called a projective R-module.

(a) If
$$\begin{tikzcd} & M \arrow[d] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$
is a diagram of R-module homomorphisms with an exact row, then there is a homomorphism $M \to A$ such that the diagram

$$\begin{tikzcd} & M \arrow[dl, dashed, "\exists"'] \arrow[d] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

commutes.

(b) The R-module M is a direct summand of a free R-module.

(c) The localization M_P is a free R_P-module for all prime ideals P of R.

Remark 1.18. We will be particularly interested in "projective modules of constant rank". The R-module M is a projective R-module of constant rank r if $M_P \cong R_P^r$ for all prime ideals P of R.

Example 1.19. Let R be the ring $\mathbb{Z}/(6\mathbb{Z})$. The R-module $M = \mathbb{Z}/(2\mathbb{Z})$ is projective but NOT of constant rank. The ring R has two prime ideals (2) and (3). The localization $M_{(2)}$ is isomorphic to $R_{(2)}$. The localization $M_{(3)}$ is zero. This example exists only because R has non-trivial idempotents: $3^2 = 3$ and $4^2 = 4$ in R.

Observation 1.20. *Let R be a commutative Noetherian ring, and $\phi : F \to G$ be a homomorphism of finitely generated free R-modules. Then* $\operatorname{coker}\phi$ *is a projective R-module of constant rank if and only if $I(\phi) = R$. Furthermore, if $I(\phi) = R$, then*

$$0 \to \ker\phi \xrightarrow{i} F \xrightarrow{\phi} G \xrightarrow{\pi} \operatorname{coker}\phi \to 0 \tag{1.20.1}$$

is a split exact sequence with $\operatorname{rank}\operatorname{coker}\phi = \operatorname{rank} G - \operatorname{rank}\phi$ *and* $\operatorname{rank}\ker\phi = \operatorname{rank} F - \operatorname{rank}\phi$.

Proof. Let $r = \operatorname{rank}\phi$.

($\Leftarrow$) Assume $I(\phi) = R$. Let P be a prime ideal of R. Observe that $\operatorname{rank}\phi_P$ is also r and that some $r \times r$ minor of ϕ_P is a unit. After a change of bases, $\phi_P = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, $G_P = (\operatorname{im}\phi)_P \oplus X$ and $F_P = Y \oplus (\ker\phi)_P$ where X is a free summand of G_P of rank $\operatorname{rank} G - \operatorname{rank}\phi$, Y is a free summand of F_P of rank $\operatorname{rank} F - \operatorname{rank}\phi$, and $\phi_P : Y \to \operatorname{im}\phi_P$ is an isomorphism. Observe that $(\operatorname{coker}\phi)_P \cong X$. Thus, $\ker\phi$ and $\operatorname{coker}\phi$ are projective R modules with constant rank and that rank is the anticipated rank. It follows that (1.20.1) is split exact.

($\Rightarrow$) Assume $\operatorname{coker}\phi$ is a projective R-module of constant rank s. Let P be a fixed prime ideal of R. It follows that (1.20.1)$_P$ is split exact:

$$\begin{tikzcd} 0 \arrow[r] & (\ker\phi)_P \arrow[r, bend left, "i_P"] & F_P \arrow[l, bend left, "\sigma_2"] \arrow[r, bend left, "\phi_P"] & G_P \arrow[l, bend left, "\sigma_1"] \arrow[r, bend left, "\pi_P"] & (\operatorname{coker}\phi)_P \arrow[l, bend left, "\sigma_0"] \arrow[r] & 0 \end{tikzcd}$$

with

$$\begin{aligned} \pi_P \circ \sigma_0 &= \operatorname{id}_{(\operatorname{coker}\phi)_P} \\ \sigma_0 \circ \pi_P + \phi_P \circ \sigma_1 &= \operatorname{id}_{G_P} \\ \sigma_1 \circ \phi_P + i_p \circ \sigma_2 &= \operatorname{id}_{F_P} \\ \sigma_2 \circ i_P \qquad\qquad &= \operatorname{id}_{(\ker\phi)_P} \end{aligned}$$

and

$$\begin{aligned} G_P &= \ker\pi_P \oplus \operatorname{im}\sigma_0 \\ F_P &= \ker\phi_P \oplus \operatorname{im}\sigma_1. \end{aligned}$$

When we record ϕ_P with respect to this direct sum decomposition, we have

$$\begin{array}{c} \ker\phi_P \\ \oplus \\ \operatorname{im}\sigma_1 \end{array} \xrightarrow{\begin{bmatrix} 0 & \cong \\ 0 & 0 \end{bmatrix}} \begin{array}{c} \ker\pi_P \\ \oplus \\ \operatorname{im}\sigma_0. \end{array}$$

The last matrix might need more justification.

The position of the nonzero component in the matrix is correct: because

$$\operatorname{im}\phi_P \circ \sigma_1 \subseteq \operatorname{im}\phi_P = \ker\pi_P.$$

The map $\operatorname{im}\sigma_1 \to \ker\pi_P$ **is onto:** If $x \in \ker\pi_P$, then

$$x = (\sigma_0 \circ \pi_P + \phi_P \circ \sigma_1)(x) = (\phi_P \circ \sigma_1)(x).$$

The map $\operatorname{im}\sigma_1 \to \ker\pi_P$ **is injective:** If $x \in \operatorname{im}\sigma_1 \cap \ker\phi_P = 0$, then x is zero.

Thus ϕ_P has rank equal to $\operatorname{rank}\ker\pi_P = \operatorname{rank} G_P - \operatorname{rank}\operatorname{im}\sigma_0 = \operatorname{rank} G - s$. In other words, ϕ has constant rank and $(I_{\operatorname{rank}\phi}(\phi))_P = R_P$. We conclude that $I(\phi) = R$.

We just used a piece of folklore: if M is a module with $M_P = 0$ for all P, then $M = 0$. $\square$

Observation 1.21. *A complex of free R-modules*

$$F \xrightarrow{\phi} G \xrightarrow{\psi} H \tag{1.21.1}$$

with $I(\phi) = I(\psi) = R$ is exact if and only if $\operatorname{rank}\phi + \operatorname{rank}\psi = \operatorname{rank} G$.

Proof. We may prove the result locally; consequently, we may assume that $(R, \mathfrak{m})$ is a local ring. Apply Observation 1.20 to write ϕ in the form

$$\begin{matrix} F' \\ \oplus \\ F'' \end{matrix} \xrightarrow{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}} \begin{matrix} G' \\ \oplus \\ G''. \end{matrix}$$

Of course, ψ is zero on G'. Apply Observation 1.20 to the restriction of ψ to G'' to see that (1.21.1) is

$$\begin{matrix} F' \\ \oplus \\ F'' \end{matrix} \xrightarrow{\begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}} \begin{matrix} G' \\ \oplus \\ G''' \\ \oplus \\ G'''' \end{matrix} \xrightarrow{\begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}} \begin{matrix} H' \\ \oplus \\ H''. \end{matrix}$$

Thus, (1.21.1) is exact if and only if $G'''' = 0$ if and only if $\operatorname{rank}\phi + \operatorname{rank}\psi = \operatorname{rank} G$. ☐

1.D. **What makes a complex exact?** I am taking this from [1, page 207] and/or [3, page 496].

Theorem 1.22. (Due to Buchsbaum and Eisenbud, see [2]**)** *Let R be a Noetherian ring and*

$$\mathbb{F}: \quad 0 \to F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \tag{1.22.1}$$

be a complex of finitely generated free R-modules. The following statements are equivalent.

(a) $\mathbb{F}$ *is acyclic*

(b) $\operatorname{rank} F_k = \operatorname{rank} f_{k+1} + \operatorname{rank} f_k$ *and either $I(\phi_k) = R$ or $I(\phi_k)$ contains a regular sequence of length k for $1 \le k \le n$.*

Example 1.23. In the syzygies talk I gave a hands-on proof that

$$0 \to R(-4)^2 \xrightarrow{\phi_2 = \begin{bmatrix} y & 0 \\ -x & y \\ 0 & -x^2 \end{bmatrix}} \begin{matrix} R(-3)^2 \\ \oplus \\ R(-2)^1 \end{matrix} \xrightarrow{\phi_1 = \begin{bmatrix} x^3 & x^2y & y^2 \end{bmatrix}} R \tag{1.23.1}$$

is a resolution. I might have applied the above Buchsbaum and Eisenbud criteria. We see that

$$\begin{aligned} \operatorname{rank} F_2 &= 2 = \operatorname{rank}\phi_2, \\ \operatorname{rank} F_1 &= 3 = 2 + 1 = \operatorname{rank}\phi_2 + \operatorname{rank}\phi_1, \end{aligned}$$

and

$$I(\phi_2) = I_2(\phi_2) = (x^3, x^2y, y^2) = I_1(\phi_1) = I(\phi_1).$$

This ideal contains x^3, y^2 which is a regular sequence of length 2. This is an alternate proof that (1.23.1) is a resolution.

Proof. (b) $\Rightarrow$ (a). Induct on the length of $\mathbb{F}$. Consider

$$0 \to F_1 \xrightarrow{f_1} F_0 \tag{1.23.2}$$

with $\operatorname{rank} F_1 = \operatorname{rank} f_1$ and the ideal generated by the maximal minors of f_1 contains a regular element. We show that f_1 is injective.

Let $r_i = \operatorname{rank} F_i$ for i equal to 0 and 1. View f_1 as an $r_0 \times r_1$ matrix. Notice that $r_1 \le r_0$. Pick r_1 rows of f_1 and consider the projection map $\operatorname{proj} : F_0 = R^{r_0} \to R^{r_1}$ which maps onto the basis elements of F_0 which correspond to the chosen rows. Let C be the classical adjoint of the chosen rows. Thus, the composition

$$R^{f_1} = F_1 \xrightarrow{f_1} F_0 \xrightarrow{\text{proj}} R^{r_1} \xrightarrow{C} R^{f_1}$$

is multiplication by the relevant maximal minor. If θ, from F_1, is in the kernel of f_1, then $I(f_1)$ times θ equals zero. Some element of $I(f_1)$ is a regular element of R. Thus, $\theta = 0$. We have shown that f_1 is an injection; thus, (1.23.2) is acyclic.

Now we assume that the result has been established for complexes of length $n-1$ and we study the complex $\mathbb{F}$ of (1.22.1). It follows from induction that

$$0 \to F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \to \dots \xrightarrow{f_2} F_1 \to \operatorname{coker}(f_2) \to 0$$

is exact. We "need only show" that the induced map

$$\operatorname{coker}(f_2) \xrightarrow{\bar{f}_1} F_0$$

is an injection. Possibly it helps to write the short exact sequence

$$0 \to \underbrace{\ker \bar{f}_1}_{\mathrm{H}_1(\mathbb{F}) = \frac{\ker f_1}{\operatorname{im} f_2}} \to \underbrace{\frac{F_1}{\operatorname{im} f_2}}_{\operatorname{coker}(f_2)} \xrightarrow{\bar{f}_1} F_0 \to 0.$$

The argument is still pretty tricky. Let H_1 denote $\mathrm{H}_1(\mathbb{F})$. We assume $\mathrm{H}_1 \neq 0$ and we deduce a contradiction. There are a few steps.

(A) We find a prime ideal Q with $\operatorname{grade}(QR_Q, (\mathrm{H}_1)_Q) = 0$, but $\operatorname{grade}(QR_Q) \neq 0$.

(B) We use the sliding hypothesis $k \le \operatorname{grade}(I(f_k))$ or $I(f_k) = R$ to split off part of $\mathbb{F}_Q$, if necessary, in order to know that the new $\mathbb{F}_Q$ stops at position $\operatorname{grade}(QR_Q)$ or less.

(C) We use $0 \to A \to B \to C \to 0$ exact implies

$$\min\{\operatorname{grade}(QR_Q, A) - 1, \operatorname{grade}(QR_Q, B)\} \le \operatorname{grade}(QR_Q, C)$$

many times to show that

$$1 \le \operatorname{grade}(QR_Q, (\operatorname{coker} f_2)_Q).$$

Once we have established (A), (B), and (C), we will have $(H_1)_Q \subseteq (\operatorname{coker} f_2)_Q$ with $\operatorname{grade}(QR_Q, (H_1)_Q) = 0$ but $1 \le \operatorname{grade}(QR_Q, (\operatorname{coker} f_2)_Q)$. This is not possible because it is impossible for every element of QR_Q to be a zero divisor on some part of $(\operatorname{coker} f_2)_Q$ and simultaneously to have some element of QR_Q to be regular on all of $(\operatorname{coker} f_2)_Q$.

Note! When $(S, \mathfrak{m})$ is a Noetherian local ring and N is a finitely generated S-module, then one often writes $\operatorname{depth} N$ in place of $\operatorname{grade}(\mathfrak{m}, N)$.

The hypothesis that $I(f_k)$ always contains a unit or a regular element forces

$$\operatorname{rank}(f_k)_P = \operatorname{rank} f_k \text{ for all } k \text{ and } I(f_k)_P = R_P \text{ for all } P \in \operatorname{Ass}(R).$$

Thus, $\mathbb{F}_P$ is split exact for $P \in \operatorname{Ass}(R)$ by Observation 1.21. It follows that $(\mathrm{H}_i)_P = 0$ for all positive i for $P \in \operatorname{Ass}(R)$; in particular $(\mathrm{H}_1)_P = 0$. It follows further that $\operatorname{ann} \mathrm{H}_1 \not\subseteq P$ for all $P \in \operatorname{Ass} R$ and by the Prime Avoidance Lemma

$$\operatorname{ann} \mathrm{H}_1 \not\subseteq \cup_{P \in \operatorname{Ass} R} P = \text{the set of zero divisors on } R.$$

In particular, there is an element in the annihilator of H_1 which is regular on R. Let Q be any prime minimal over $\operatorname{ann} \mathrm{H}_1$. It follows automatically, that $Q \in \operatorname{Ass} \mathrm{H}_1$; in particular $QR_Q \in \operatorname{Ass}(\mathrm{H}_1)_Q$ and $\operatorname{grade}(QR_Q, (\mathrm{H}_1)_Q) = 0$. On the other hand, Q contains an element which is regular on R. Localization is exact; so, $1 \le \operatorname{grade}(QR_Q)$.

Let $m = \operatorname{grade}(QR_Q)$. If $m \le n$, then the hypothesis about $I(f_k)$ ensures that

$$I(f_k)_Q = R_Q$$

for $m + 1 \le k \le n$; and therefore,

$$0 \to (F_n)_Q \to \cdots \to (F_m)_Q \to (\operatorname{coker} f_{m+1})_Q \to 0$$

is split exact and $(\operatorname{coker} f_{m+1})_Q$ is a projective (hence free) R_Q-module. We now have

$$0 \to (\operatorname{coker} f_{m+1})_Q \to (F_{m-1})_Q \to \cdots \to (F_1)_Q \to (\operatorname{coker} f_2)_Q \to 0$$

is exact. In any event we have:

$$0 \to F'_\ell \to \dots F'_1 \to (\operatorname{coker} f_2)_Q \to 0$$

an exact sequence of R_Q-modules with F'_i a free R_Q-module and $\ell \le m$. Apply the ABC result to the short exact sequences

$$\begin{gathered} 0 \to F'_\ell \to F'_{\ell-1} \to X_{\ell-1} \to 0 \\ 0 \to X_{\ell-1} \to F'_{\ell-2} \to X_{\ell-2} \to 0 \\ 0 \to X_{\ell-2} \to F'_{\ell-3} \to X_{\ell-3} \to 0 \\ \vdots \\ 0 \to X_2 \to F'_1 \to (\operatorname{coker} f_2)_Q \to 0 \end{gathered}$$

to learn

$$\begin{gathered} m - 1 \le \operatorname{depth} X_{\ell-1} \\ m - 2 \le \operatorname{depth} X_{\ell-2} \\ m - 3 \le \operatorname{depth} X_{\ell-3} \end{gathered}$$

$$\vdots$$

$$1 \le m - \ell + 1 \le \operatorname{depth}(\operatorname{coker} f_2)_Q.$$

This completes the proof of (C) and therefore the proof of (b) ⇒ (a).

(a) ⇒ (b).

Suppose $\mathbb{F}$ from (1.22.1):

$$\mathbb{F}: \quad 0 \to F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \to \cdots \to F_1 \xrightarrow{f_1} F_0$$

is acyclic. We must show that

$$\operatorname{rank} F_k = \operatorname{rank} f_k + \operatorname{rank} f_{k+1} \quad \text{for} \quad 1 \le k \le n \quad \text{and}$$

$$k \le \operatorname{grade} I(f_k) \quad \text{or} \quad I(f_k) = R \quad \text{for} \quad 1 \le k \le n.$$

We first show that the rank of the matrices f_i does not drop when we localize. We do this by showing that each $I(f_k)$ contains either a unit or a regular element. We prove this by inverting all non-zero divisors and showing that the resulting complex is split exact. We will use the Auslander-Buchsbaum formula:

Theorem 1.24. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module of finite projective dimension, then*

$$\operatorname{depth} R = \operatorname{depth} M + \operatorname{pd}_R M.$$

The proof is given in 3.3.

Let $M = \mathrm{H}_0(\mathbb{F})$. (So, $\mathbb{F}$ is a resolution of M.)

Let S be the set of non-zero divisors of R. Consider the ring $S^{-1}(R)$. Observe that $S^{-1}(\mathbb{F})$ is a resolution of $S^{-1}(M)$ by free $S^{-1}(R)$-modules. The maximal ideals of $S^{-1}(R)$ are the maximal ideals of $\operatorname{Ass}(R)$. (Recall that $\operatorname{Ass}(R)$ is a finite set of ideals. There could be chains of such ideals, ordered by inclusion. We are only taking the biggest ideal in such chains.) Each such maximal ideal consists of zero divisors on $S^{-1}(R)$; that is, $\operatorname{depth}(S^{-1}(R))_P = 0$ (Of course, $(S^{-1}(R))_P = R_P$.) At any rate, the Auslander-Buchsbaum Theorem yields that M_P is projective (hence free). Thus, $(S^{-1}(\mathbb{F}))_P$ is split exact for each maximal ideal P of $S^{-1}(R)$ (This is the property of mapping onto a projective module.) and $(S^{-1}(M))_P$ is a free $(S^{-1}(R))_P$ with a rank that does not depend on P (The answer is the alternating sum of the Betti numbers in the resolution). Apply Observation 1.20 and then Observation 1.21 to see that $I(S^{-1}(f_k)) = S^{-1}(R)$ for each k and $\operatorname{rank} S^{-1}(f_k) + \operatorname{rank} S^{-1}(f_{k+1}) = \operatorname{rank} S^{-1}F_k$. Thus, every $I(f_k)$ contains a non-zero divisor – so the rank of f_k is equal to the rank of any localization of f_k. We learn in particular, that $\operatorname{rank} f_k + \operatorname{rank} f_{k+1} = \operatorname{rank} F_k$ for each k. We need only show the condition about each $I(f_k)$.

Fix k with $I(f_k) \ne R$. Let $\underline{x}$ be a maximal regular sequence on R in $I(f_k)$. Pick $P \in \operatorname{Ass} R/(\underline{x})$ with $I(f_k) \subseteq P$. We can do this:

$$I(f_k) \subseteq \text{ZeroDivisors}(R/(\underline{x})) = \cup_{P \in \operatorname{Ass} R/(\underline{x})} P.$$

It follows that $\operatorname{grade} PR_P = \operatorname{grade} I_k(f_k)$. Localize $\mathbb{F}$ at P. The ideal $I(f_k)_P$ is not equal to R_P; thus, by Observation 1.20, $\operatorname{coker}(f_k)_P$ is not free. (Keep in mind that $\operatorname{coker}(f_k)_P$

is the $(k-1)$-st syzygy of M_P. If the $(k-1)$-st syzygy of some module is not projective, then $k \le \operatorname{pd}(\text{the module})$.) It follows that $k \le \operatorname{pd}_{R_P} M_P$ and hence, by the Auslander-Buchsbaum formula, $k \le \operatorname{depth} R_P = \operatorname{grade} I_k(f_k)$. ☐

Remarks 1.25. It is often possible to make grade calculations.

(a) $\operatorname{grade} I = \operatorname{grade} \sqrt{I}$ This is immediate from a more general version of Theorem 1.13 where R/I is replaced with any finitely generated R-module N with the property that $N_P \neq 0 \iff I \subseteq P$. The same proof works with virtually no change. See [1, pg 206] or prove that

$$\operatorname{grade} I = \min\{\operatorname{depth} R_P \mid P \in \operatorname{Spec} R \text{ and } I \subseteq P\};$$

see [8, pg. 105].

(b) In a Cohen-Macaulay local ring $\operatorname{grade} I = \operatorname{height} I$. Recall that "grade" is a homological measure of the "size" of I and "height" is a geometric measure of "size". (See [8, Thm. 31 on pg 108].)

(c) "Often" $\operatorname{grade}(I+x)/(x) \le \operatorname{grade} I$. [1] The grade on the left is calculated in the ring R/x, the grade on the right is calculated in the ring R, I is an ideal of R and x is an element of R. For example, in the "Syzygies" Colloquium, I claimed that

$$0 \to R(-3)^2 \xrightarrow{f_2 = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \\ x_3 & x_4 \end{bmatrix}} R(-2)^3 \xrightarrow{f_1 = \left[\begin{vmatrix} x_2 & x_3 \\ x_3 & x_4 \end{vmatrix},\; -\begin{vmatrix} x_1 & x_2 \\ x_3 & x_4 \end{vmatrix},\; \begin{vmatrix} x_1 & x_2 \\ x_2 & x_3 \end{vmatrix}\right]} R$$

is a resolution when $R = \boldsymbol{k}[x_1, x_2, x_3, x_4]$. Of course, I applied the BE-criteria, but how did I know that

$$2 \le \operatorname{grade} I_2(f_2)?$$

Well, if I set $x_1 = 0$, then $I_2(f_2)$ is $(x_2x_4 - x_3^2, x_2x_3, x_2^2)$ and this ideal has radical (x_2, x_3). So, we have identified homogeneous elements of positive degree $x_2x_4 - x_3^2, x_1x_3 - x_2^2, x_1$ in R, with $x_2x_4 - x_3^2, x_1x_3 - x_2^2$ in I so that $\overline{x_2x_4 - x_3^2}, \overline{x_1x_3 - x_2^2}$ is a regular sequence in $\bar{R} = R/(x_1)$. We conclude that $2 \le \operatorname{grade} I$.

---

[1] A correct argument would use two facts about regular sequences. See, for example, [9, Thm. 16.1 and the Corollary on page 127]. These facts are well-known result. They can be found elsewhere, also. Fact 1. If R is a commutative Noetherian ring, $f_1, \dots f_n$ is a regular sequence on R, and $a_1, \dots, a_n$ are non-negative integers, then $f_1^{a_1}, \dots f_n^{a_n}$ is a regular sequence on R. One consequence of this fact is that $\operatorname{grade} I = \operatorname{grade} \sqrt{I}$ for any proper ideal I in R. Fact 2. If R is a local ring then any permutation of a regular sequence is also a regular sequence. Similarly, if $R = \bigoplus_{0 \le i} R_i$ is a commutative Noetherian ring, then any permutation of a regular sequence of homogeneous elements of positive degree is also a regular sequence. So the moral is: If $(R, \mathfrak{m})$ is a local ring, $x \in \mathfrak{m}$, and I is an ideal of R, then $\operatorname{grade} \frac{\sqrt{(I,x)}}{(x)} \le \operatorname{grade} I$. Similarly, if $R = \bigoplus_{0 \le i} R_i$ is a commutative Noetherian ring, $f_1, \dots f_n, x$ are homogeneous elements of positive degree in R, with $\bar{f}_1, \dots \bar{f}_n$ a regular sequence in $\bar{R} = R/(x)$, then $f_1, \dots f_n$ is a regular sequence in R.

2. THE HILBERT-BURCH THEOREM

Theorem 2.1. *Let R be a local ring and I be a non-zero ideal of R so that R/I has free resolution*

$$0 \to F_2 \xrightarrow{\phi_2} R^n \xrightarrow{\phi_1} R,$$

then $F_2 = R^{n-1}$ and there exists a regular element r in R such that

$$\phi_1 = r \begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix},$$

where X_i is $(-1)^{i+1}$ times the determiant of the matrix for ϕ_2 with row i deleted.

The rank of F_2 is given by the following more general result. The critical fact in this Lemma is the statement about $\operatorname{ann} M$ containing a regular element. This result is originally due to Auslander-Buchsbaum with a different argument.

Lemma 2.2. *Let R be a commutative Noetherian ring, M be a finitely generated R-module, and*

$$\mathbb{F} : 0 \to F_n \xrightarrow{f_n} \dots \xrightarrow{f_1} F_0$$

be a finite free resolution of M. If $\operatorname{ann} M \neq 0$, then there is an element in $\operatorname{ann} M$ which is regular on R and $\sum_i (-1)^i \operatorname{rank} F_i = 0$.

Proof. WMACE guarantees that there is an element s in $I(f_1)$ with s regular on R. (This s might be a unit; but that does not bother us.) We look at $S^{-1}M$ over the ring $S^{-1}R$, for $S = \{1, s, s^2, \dots\}$. Notice that $S^{-1}M$ has a non-zero annihilator as an $S^{-1}R$-module. The ideal of maximal minors of the presentation matrix for $S^{-1}M$ as an $S^{-1}R$-module contains a unit; so $S^{-1}M$ is a projective $S^{-1}R$-module of constant rank by an earlier result.

We claim that $S^{-1}M$ must be zero. Assume not, reach a contradiction. If $S^{-1}M \neq 0$, then $(S^{-1}M)_P = (S^{-1}R)_P^{\#}$ for some $\#$ which is independent of $P \in \operatorname{Spec}(S^{-1}R)$. We have a non-zero element θ of $S^{-1}R$ which annihilates $S^{-1}M$. Thus, θ annihilates $(S^{-1}M)_P = (S^{-1}R)_P^{\#}$ for all $P \in \operatorname{Spec}(S^{-1}R)$; hence $\theta = 0$ in $(S^{-1}R)_P$ for all $P \in \operatorname{Spec}(S^{-1}R)$; and therefore, $\theta = 0$ in $S^{-1}R$. This is a contradiction. Hence $S^{-1}M$ is zero. We have shown, in particular, that $\operatorname{ann} M$ contains a regular element.

Now take any prime P missing s. Observe $M_p = 0$ and $(f_1)_P = [I|0]$. In particular, $\operatorname{rank} f_1$ (which is independent of P) is equal to $\operatorname{rank} F_0$. Now we are finished:

$$\begin{array}{rrr}
 & \operatorname{rank} f_n & = \operatorname{rank} F_n \\
\operatorname{rank} f_n & + \operatorname{rank} f_{n-1} & = \operatorname{rank} F_{n-1} \\
\vdots & & \\
\operatorname{rank} f_2 & + \operatorname{rank} f_1 & = \operatorname{rank} F_1 \\
\operatorname{rank} f_1 & & = \operatorname{rank} F_0
\end{array}$$

So, the sum with alternating signs is indeed zero. □

Proof of the Hilbert-Burch Theorem. We know from Lemma 2.2 that $F_2 = R^{n-1}$. It follows that f_2 is a $n \times (n-1)$ matrix. Let $\begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}$ be the row vector of signed maximal minors of f_2. We want to compare the two complexes

$$0 \to R^{n-1} \xrightarrow{f_2} R^n \xrightarrow{f_1} R$$

and

$$0 \to R^{n-1} \xrightarrow{f_2} R^n \xrightarrow{\begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}} R.$$

I do not know how to do that. So, I turn the complexes around and look at

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "f_1^{\mathrm{T}}"] \arrow[d, dashed] & R^n \arrow[r, "f_2^{\mathrm{T}}"] \arrow[d, "="] & R^{n-1} \arrow[r] \arrow[d, "="] & \operatorname{coker} f_2^{\mathrm{T}} \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & R \arrow[r, "\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}"'] & R^n \arrow[r, "f_2^{\mathrm{T}}"] & R^{n-1} \arrow[r] & \operatorname{coker} f_2^{\mathrm{T}} \arrow[r] & 0
\end{tikzcd}$$

The bottom complex is exact. The top complex is exact. There is a comparison map. The map at the far right is multiplication by some element r. We have shown that

$$f_1^{\mathrm{T}} = r \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}.$$

3. THE AUSLANDER-BUCHSBAUM FORMULA

The proof of the final direction of the WMACE result heavily uses the Auslander-Buchsbaum formula. So, lets prove this formula.

Theorem 3.1. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local ring and M be a finitely generate R-module of finite projective dimension. Then*

$$\operatorname{depth} R = \operatorname{depth} M + \operatorname{pd}_R M.$$

Remarks 3.2. (a) If the ambient ring is local with maximal ideal $\mathfrak{m}$, then $\operatorname{depth} X = \operatorname{grade}(\mathfrak{m}, X)$.

(b) If M is a non-zero finitely generated module over a local ring then $\mathfrak{m}M \neq M$ happens automatically because of Nakyama's Lemma. (Nakayama's Lemma is cool. We prove it in 3.4.)

(c) Projective dimension is a well-defined notion. If some projective resolution of M has length n, then every other resolution of M by projective R-modules can be modified to have length less than or equal to n. This is due to Schanuel's Lemma: If

$$0 \to K \to P \to M \to 0 \quad \text{and} \quad 0 \to K' \to P' \to M \to 0$$

are exact with P and P' projective, then

$$P \oplus K' \cong P' \oplus K;$$

and similarly, if

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0 \quad \text{and} \quad 0 \to K' \to P'_{n-1} \to \cdots \to P'_1 \to P'_0 \to M \to 0$$

are exact with P_i and P'_i projective, then

$$K \oplus P'_{n-1} \oplus P_{n-2} \oplus \cdots \oplus \boxed{}_0 \cong K' \oplus P_{n-1} \oplus P'_{n-2} \oplus \cdots \oplus \boxed{}_0;$$

hence, if one of K or K' is projective, then both are projective. We prove Schanuel's Lemma in 3.6.

(d) If $(R, \mathfrak{m}, \boldsymbol{k})$ is a local ring, M is a finitely generated R-module, and F is a resolution of M by free R-modules, then F is <u>minimal</u> if every entry of each matrix in F is in $\mathfrak{m}$. A minimal resolution always has length equal to $\operatorname{pd}_R M$. (It also has many other minimality properties.) One can always build a minimal resolution – just take take a minimal generating set for each syzygy. (One can use Nakayama's Lemma to find a minimal generating set for a given syzygy.)

(e) My proof of the Auslander-Buchsbaum formula will use Tor. The functor Tor is "just like" the functor Ext; except Tor is easier. One only uses projective resolutions (and never any injective resolutions) and the functors $M \otimes -$ and $- \otimes N$ are both covariant (neither is contravariant). But the basic idea is still the same. Once one makes sense out of either one; it becomes easier to make sense of the other.

How to compute Tor. Let M and N be R-modules; P be a projective resolution of M and Q be a projective resolution of N. Then

$$\mathrm{H}_i(P \otimes_R N) \quad \text{and} \quad \mathrm{H}_i(M \otimes_R Q)$$

are isomorphic and both are called $\operatorname{Tor}_i^R(M, N)$.

The long exact sequence of homology. If $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules and M is an R-module, then apply $M \otimes_R -$ to the short exact sequence to obtain the long exact sequence of homology:

$$\cdots \to \operatorname{Tor}_2(M,C) \to \operatorname{Tor}_1(M,A) \to \operatorname{Tor}_1(M,B) \to \operatorname{Tor}_1(M,C) \to M\otimes_R A \to M\otimes_R B \to M\otimes_R C \to 0$$

3.3. Prove the Auslander-Buchsbaum Formula. The proof is by induction on $\operatorname{depth} R$.

First assume that $\operatorname{depth} R = 0$. We will prove that M is projective (and hence free). This will make both $\operatorname{depth} M$ and $\operatorname{pd}_R M$ both be 0. The hypothesis $\operatorname{depth} R = 0$ guarantees that every element of $\mathfrak{m}$ is a zero-divisor on R; hence $\mathfrak{m} \subseteq \cup_{P\in \operatorname{Ass} R} P$. The prime avoidance lemma then forces $\mathfrak{m}$ to be an associated prime of R. In other words, there is an embedding of $\boldsymbol{k}$ into R. Let C be the cokernel. We have produced a short exact sequence:

$$0 \to \boldsymbol{k} \to R \to C \to 0. \tag{3.3.1}$$

Let $p = \operatorname{pd}_R M$. Keep in mind that

$$\operatorname{Tor}_i(M, \boldsymbol{k}) = \boldsymbol{k}^{i\text{-th Betti number of } M} \neq 0 \quad \text{for } 0 \le i \le p$$

(Apply $- \otimes \boldsymbol{k}$ to a minimal resolution of M; the differentials become zero; so the kernels are everything and the images are zero.) and

$$\operatorname{Tor}_i(M, -) = 0 \quad \text{for } p+1 \le i.$$

On the other hand when $M \otimes_R -$ is applied to (3.3.1), one obtains

$$\operatorname{Tor}_{i+1}(M, C) \cong \operatorname{Tor}_i(M, \boldsymbol{k}) \quad \text{for } 1 \le i$$

(because $\operatorname{Tor}_i^R(R, -) = 0$ for positive i). We know that $\operatorname{Tor}_{p+1}(M, C)$ is not isomorphic to $\operatorname{Tor}_p(M, \boldsymbol{k})$. It follows that $p = 0$, as desired!

Now assume that $1 \le \operatorname{depth} R$. We treat two cases separately. Either $1 \le \operatorname{depth} M$ or $0 = \operatorname{depth} M$.

Assume $\operatorname{depth} R$ and $\operatorname{depth} M$ are both positive. This means that there is an element x in $\mathfrak{m}$ which is regular on both R and $\mathfrak{m}$. (This assertion requires a moments thought. The assumption guarantees that $\mathfrak{m}$ is not an associated prime of R or M. The prime avoidance lemma guarantees that

$$\mathfrak{m} \not\subseteq \left(\bigcup_{P\in \operatorname{Ass} R} P \right) \bigcup \left(\bigcup_{P \in \operatorname{Ass} M} P \right).$$

Take x from $\mathfrak{m}$ but not from any of the associated primes.) It is clear that $\operatorname{depth} R/(x) = \operatorname{depth} R - 1$ and $\operatorname{depth} M/(x)M = \operatorname{depth} M - 1$. We finish the argument by showing that $\operatorname{pd}_R M = \operatorname{pd}_{R/x} M/(x)M$. Of course, this is easy. Let F be a minimal resolution of M

by free R-modules. We want to show that $F \otimes R/(x)$ is a resolution of $M/(x)M$ by free-$R/(x)$ modules. (This resolution will automatically be minimal.) That is, we want to show that $\operatorname{Tor}_i^R(M, R/(x)) = 0$ for $1 \le i$. The fact that $\operatorname{pd}_R R/(x) = 1$ guarantees that $\operatorname{Tor}_i^R(M, R/(x)) = 0$ for $2 \le i$. The fact that x is regular on M takes care of Tor_1.

Assume $1 \le \operatorname{depth} R$ and $0 = \operatorname{depth} M$. Let K be the first syzygy of M. Observe that $\operatorname{depth} K = 1$. (We could do the ABC Lemma twice, once for each inequality.) Instead, we just do the argument directly. Apply $\operatorname{Hom}_R(\boldsymbol{k}, -)$ to the short exact sequence

$$0 \to K \to R^t \to M \to 0$$

to obtain the long exact sequence

$$0 \to \operatorname{Hom}_R(\boldsymbol{k}, K) \to \underbrace{\operatorname{Hom}_R(\boldsymbol{k}, R^t)}_{0} \to \underbrace{\operatorname{Hom}_R(\boldsymbol{k}, M)}_{\neq 0} \to \operatorname{Ext}^1_R(\boldsymbol{k}, K) \to \operatorname{Ext}^1_R(\boldsymbol{k}, R^t) \to \operatorname{Ext}^1_R(\boldsymbol{k}, M)$$

Thus, $\operatorname{Hom}_R(\boldsymbol{k}, K) = 0$ and $\operatorname{Ext}^1_R(\boldsymbol{k}, K) \neq 0$. It follows that $\operatorname{depth} K = 1$. We may apply the earlier case of the present argument to see that

$$\operatorname{depth} R = \operatorname{pd}_R K + \operatorname{depth} K = (\operatorname{pd}_R M - 1) + (\operatorname{depth} M + 1). \quad \square$$

I am not sure that it is essential that we prove Nakyama's Lemma; but it is such an empowering result and I promised to prove it.

Lemma 3.4. (Nakyama's Lemma) *Let M be a finitely generated module over the local ring $(R, \mathfrak{m})$ and let $m_1, \dots, m_n$ be elements of M. Then*

$$m_1, \dots, m_n \text{ generate } M \iff \bar{m}_1, \dots, \bar{m}_n \text{ generate } M/\mathfrak{m}M,$$

where $^-$ represents the image in $M/\mathfrak{m}M$.

Remark 3.5. The result is really cool because $M/\mathfrak{m}M$ is a finite dimensional vector space; hence, there is a well-defined notion of basis for $M/\mathfrak{m}M$. As a consequence, there is a well-defined notion of "the minimal number of generators for M". Any set of elements $m_1, \dots, m_i$ in M with $\bar{m}_1, \dots, \bar{m}_i$ linearly independent can be extended to a minimal generating set for M. Any generating set for M contains a minimal generating set.

Proof of Nakyama's Lemma. One direction is obvious. We prove the other direction. We assume $\bar{m}_1, \dots, \bar{m}_n$ generate $M/\mathfrak{m}M$. It follows that $(m_1, \dots, m_n)R + \mathfrak{m}M = M$; hence,

$$M/(m_1, \dots, m_n)R$$

is killed by $\mathfrak{m}$. It suffices to show that if N is a finitely generated R module with $N = \mathfrak{m}N$, then $N = 0$.

Let $\theta_1, \dots, \theta_s$ generate N. Identify a matrix A with entries in $\mathfrak{m}$ and

$$\begin{bmatrix} \theta_1 \\ \vdots \\ \theta_s \end{bmatrix} = A \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_s \end{bmatrix}.$$

So,

$$0 = (I - A) \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_s \end{bmatrix}.$$

Multiply both sides of the equation on the left by the classical adjoint of $I - A$ to learn that

$$0 = \det(I - A) \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_s \end{bmatrix}.$$

Observe $\det(I - A) = 1 +$ an element of $\mathfrak{m}$ and this is a unit of R; hence each generator θ_i of N is zero. It follows that $N = 0$. $\square$

3.6. Proof of Schanuel's Lemma If

$$0 \to K \to P \xrightarrow{\pi} M \to 0 \quad \text{and} \quad 0 \to K' \to P' \xrightarrow{\pi'} M \to 0$$

are exact with P and P' projective, then

$$P \oplus K' \cong P' \oplus K.$$

Proof. Let $X = \{(p, p') \in P \oplus P' \mid \pi(p) = \pi(p')\}$. Observe that

$$0 \to K' \to X \to P \to 0 \quad \text{and} \quad 0 \to K \to X \to P' \to 0$$

are both split exact sequences. $\square$

If

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \xrightarrow{\pi} M \to 0 \quad \text{and} \quad 0 \to K' \to P'_{n-1} \to \cdots \to P'_1 \to P'_0 \xrightarrow{\pi'} M \to 0$$

are exact with P_i and P'_i projective, then

$$K \oplus P'_{n-1} \oplus P_{n-2} \oplus \cdots \oplus \boxed{}_0 \cong K' \oplus P_{n-1} \oplus P'_{n-2} \oplus \cdots \oplus \boxed{}_0;$$

Proof. The first case shows that

$$\ker \pi \oplus P'_0 \cong \ker \pi' \oplus P_0.$$

Thus,

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \oplus P'_0 \to \ker \pi \oplus P'_0 \to 0$$

and

$$0 \to K' \to P'_{n-1} \to \cdots \to P'_1 \oplus P_0 \to \ker \pi' \oplus P_0 \to 0$$

are exact with $\ker \pi \oplus P'_0 \cong \ker \pi' \oplus P_0$. The argument is finished by induction. $\square$

3.7. Finish the proof of WMACE. Let R be a commutative Noetherian ring and

$$F: \quad 0 \to F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} F_0$$

be an acyclic complex of finitely generated free R-modules. We prove that

$$\operatorname{rank} F_k = \operatorname{rank} f_k + \operatorname{rank} f_{k+1} \quad \text{for} \quad 1 \le k \le n \quad \text{and}$$

$$k \le \operatorname{grade} I(f_k) \quad \text{or} \quad I(f_k) = R \quad \text{for} \quad 1 \le k \le n.$$

We first show that the rank of the matrices f_i does not drop when we localize. We do this by showing that each $I(f_k)$ contains either a unit or a regular element. We prove this by inverting all non-zero divisors and showing that the resulting complex is split exact. Let $M = \mathrm{H}_0(\mathbb{F})$. (So, $\mathbb{F}$ is a resolution of M.)

Let S be the set of non-zero divisors of R. Consider the ring $S^{-1}(R)$. Observe that $S^{-1}(\mathbb{F})$ is a resolution of $S^{-1}(M)$ by free $S^{-1}(R)$-modules. The maximal ideals of $S^{-1}(R)$ are the maximal ideals of $\operatorname{Ass}(R)$. (Recall that $\operatorname{Ass}(R)$ is a finite set of ideals. There could be chains of such ideals, ordered by inclusion. We are only taking the biggest ideal in such chains.) Each such maximal ideal consists of zero divisors on $S^{-1}(R)$; that is, $\operatorname{depth}(S^{-1}(R))_P = 0$ (Of course, $(S^{-1}(R))_P = R_P$.) At any rate, the Auslander-Buchsbaum Theorem yields that M_P is projective (hence free). Thus, $(S^{-1}(\mathbb{F}))_P$ is split exact for each maximal ideal P of $S^{-1}(R)$ (This is the property of mapping onto a projective module.) and $(S^{-1}(M))_P$ is a free $(S^{-1}(R))_P$ with a rank that does not depend on P (The rank of $(S^{-1}(M))_P$ is the alternating sum of the ranks of the F_i.) Apply Observation 1.20 and then Observation 1.21 to see that $I(S^{-1}(f_k)) = S^{-1}(R)$ for each k and $\operatorname{rank} S^{-1}(f_k) + \operatorname{rank} S^{-1}(f_{k+1}) = \operatorname{rank} S^{-1}F_k$. Thus, every $I(f_k)$ contains a non-zero divisor – so the rank of f_k is equal to the rank of any localization of f_k. We learn in particular, that $\operatorname{rank} f_k + \operatorname{rank} f_{k+1} = \operatorname{rank} F_k$ for each k. We need only show the grade condition about each $I(f_k)$.

Fix k with $I(f_k) \neq R$. Let $\underline{x}$ be a maximal regular sequence on R in $I(f_k)$. Pick $P \in \operatorname{Ass} R/(\underline{x})$ with $I(f_k) \subseteq P$. We can do this:

$$I(f_k) \subseteq \text{ZeroDivisors}(R/(\underline{x})) = \cup_{P \in \operatorname{Ass} R/(\underline{x})} P.$$

It follows that $\operatorname{grade} PR_P = \operatorname{grade} I_k(f_k)$. Localize $\mathbb{F}$ at P. The ideal $I(f_k)_P$ is not equal to R_P; thus, by Observation 1.20, $\operatorname{coker}(f_k)_P$ is not free. (Keep in mind that $\operatorname{coker}(f_k)_P$ is the $(k-1)$-st syzygy of M_P. If the $(k-1)$-st syzygy of some module is not projective, then $k \leq \operatorname{pd}(\text{the module})$.) It follows that $k \leq \operatorname{pd}_{R_P} M_P$ and hence, by the Auslander-Buchsbaum theorem, $k \leq \operatorname{depth} R_P = \operatorname{grade} I_k(f_k)$.

4. THE EAGON-NORTHCOTT COMPLEX AND GENERALIZATIONS OF THE EAGON-NORTHCOTT COMPLEX.

Goal 4.1. *Let R be a commutative Noetherian ring, F and G be free R-modules of ranks f and g, respectively, with $g \le f$, and $\phi : F \to G$ be an R-module homomorphism. We want to record a family of complexes $\{\mathcal{C}^q \mid q \in \mathbb{Z}\}$ such that*

(a) $\mathrm{H}_0(\mathcal{C}^0) = R/I_g(\phi)$*, and if* $\operatorname{grade} I_g(\phi)$ *is large enough, then* $\mathcal{C}^0$ *is a resolution,*
(b) $\mathrm{H}_0(\mathcal{C}^1) = \operatorname{coker} \phi$*, and if* $\operatorname{grade} I_g(\phi)$ *is large enough, then* $\mathcal{C}^1$ *is a resolution,*
(c) *the dual of each complex in* $\{\mathcal{C}^q\}$ *is also in* $\{\mathcal{C}^q\}$*,*
(d) *if* $\mathcal{C}^q$ *has the same length as* $\mathcal{C}^0$ *and* $\operatorname{grade} I_g(\phi)$ *is large enough, then* $\mathcal{C}^q$ *and* $(\mathcal{C}^q)^*$ *are both acyclic,*
(e) *if* $-1 \le q$ *and appropriate grade condtions are satisfied by the Fitting ideals of* ϕ *(these are the ideals* $I_i(\phi)$ *with* $1 \le i \le g$*), then* $\mathcal{C}^q$ *is acyclic.*

Here is some history and some references. The complex $\mathcal{C}^0$ is the Eagon-Northcott complex. The complex $\mathcal{C}^1$ is the Buchsbaum-Rim complex. I refer to the family $\{\mathcal{C}^q\}$ as the family of generalized Eagon-Northcott complexes. (I think that when I was younger, I did not say "generalized"; but my friend and collaborator Bernd Ulrich convinced me that me that "generalized Eagon-Northcott complexes" is the correct name.) A very pretty presentation of these complexes may be found in [3, Appendix A2.6, especially page 595]. Bernd and I wrote a paper [6] that described a family of complexes (which we called $\{\mathcal{D}^q\}$) with properties similar to the properties of $\{\mathcal{C}^q\}$. A preliminary draft of [6] included a section on the family $\{\mathcal{C}^q\}$. That section did not make it into print; nonetheless, it is available on the class website under the name "MemoirWithBerndExpandedVersion.pdf". (See section 2.) Recently [5], I considered yet another family of complexes with similar properties; once again this paper recalls the properties of $\{\mathcal{C}^q\}$.

Remarks 4.2. (a) One considers a **family** of complexes, rather than **one** complex because it is easier to figure out the pattern of how to build the complex if one has a bunch of complexes to consider, rather than only one complex.
(b) One can prove acyclicity by induction when one has a family. (There is a decent chance that the mapping cone of two adjacent members of a family give rise to one member of the family built with larger data.)
(c) If the family includes the duals of the complexes of interest, then one can work from the front and back simultaneously!
(d) I can tell you $\mathrm{H}^0(\mathcal{C}^q)$ for $-1 \le q$. Indeed,

$$\mathrm{H}_0(\mathcal{C}^q) = \mathrm{Sym}_q(\operatorname{coker} \phi), \quad \text{for } 1 \le q.$$

Definition 4.3. Let R be a commutative ring and M be an R-module. The symmetric algebra of M is a commutative R-algebra $\mathrm{Sym}(M)$ together with and R-module homomorphism $M \xrightarrow{i} \mathrm{Sym}(M)$ which satisfies the following universal mapping property: Whenever S is a commutative R-algebra and $M \xrightarrow{f} S$ is an R-module homomorphism, then there exits a unique R-algebra homomorphism $\tilde{f}$ such that

$$\begin{tikzcd}
\mathrm{Sym}(M) \arrow[dr, dashed, "\exists! \tilde{f}"] & \\
M \arrow[u, "i"] \arrow[r, "f"'] & S
\end{tikzcd}$$

commutes.

The symmetric module always exists. It is the tensor algebra

$$T(M) = R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \dots$$

modded out by the two-sided ideal generated by $\{m \otimes m' - m' \otimes m \mid m, m' \in M\}$.

Notice that $\mathrm{Sym}(M)$ is automatically a graded R-module; because $T(M)$ is a graded R-algebra and one is modding out by a homogeneous ideal.

In practice, $\mathrm{Sym}(M)$ is easy to deal with. If R is Noetherian, M is finitely generated, and

$$R^s \xrightarrow{A} R^t \to M \to 0$$

is exact, then

$$\mathrm{Sym}(M) = R[T_1, \dots, T_t]/([T_1, \dots, T_t]A).$$

Of course, if M is a free module of rank t, then $\mathrm{Sym}(M)$ is the polynomial ring in t variables with coefficients from R and $\mathrm{Sym}_q(M)$ is the free R-module on the monomials of degree q in t variables.

The generalized Eagon-Northcott complexes associated to the R-module homomorphism $\phi : F \to G$, where F and G are free R-modules of finite rank f and g respectively and $g \leq f$.

$$
\begin{array}{rlrrrr}
 & & \text{position } f-g+1 & \text{position } f-g & \text{position } 1 & \text{position } 0 \\
\vdots & & & & & \\
\mathcal{C}_\phi^{-1}: & 0 \to & \bigwedge^f F \otimes \bigwedge^g G^* \otimes D_{f-g+1}(G^*) \xrightarrow{\eta_\phi} & \bigwedge^{f-1} F \otimes \bigwedge^g G^* \otimes D_{f-g}(G^*) \xrightarrow{\eta_\phi} \dots \xrightarrow{\eta_\phi} & \bigwedge^g F \otimes \bigwedge^g G^* \otimes D_1(G^*) \xrightarrow{\eta_\phi} & \bigwedge^{g-1} F \otimes \bigwedge^g G^* \otimes D_0(G^*) \to 0 \\
\mathcal{C}_\phi^{0}: & 0 \to & \bigwedge^f F \otimes \bigwedge^g G^* \otimes D_{f-g}(G^*) \xrightarrow{\eta_\phi} & \bigwedge^{f-1} F \otimes \bigwedge^g G^* \otimes D_{f-g-1}(G^*) \xrightarrow{\eta_\phi} \dots \xrightarrow{\eta_\phi} & \bigwedge^g F \otimes \bigwedge^g G^* \otimes D_0(G^*) \xrightarrow{\bigwedge^g \phi} & \bigwedge^0 F \otimes \mathrm{Sym}_0\, G \to 0 \\
\mathcal{C}_\phi^{1}: & 0 \to & \bigwedge^f F \otimes \bigwedge^g G^* \otimes D_{f-g-1}(G^*) \xrightarrow{\eta_\phi} & \bigwedge^{f-1} F \otimes \bigwedge^g G^* \otimes D_{f-g-2}(G^*) \xrightarrow{\eta_\phi} \dots \xrightarrow{\bigwedge^g \phi} & \bigwedge^1 F \otimes \mathrm{Sym}_0\, G \xrightarrow{\mathrm{Kos}_\phi} & \bigwedge^0 F \otimes \mathrm{Sym}_1\, G \to 0 \\
\mathcal{C}_\phi^{2}: & 0 \to & \bigwedge^f F \otimes \bigwedge^g G^* \otimes D_{f-g-2}(G^*) \xrightarrow{\eta_\phi} & \bigwedge^{f-1} F \otimes \bigwedge^g G^* \otimes D_{f-g-3}(G^*) \xrightarrow{\eta_\phi} \dots \xrightarrow{\mathrm{Kos}_\phi} & \bigwedge^1 F \otimes \mathrm{Sym}_1\, G \xrightarrow{\mathrm{Kos}_\phi} & \bigwedge^0 F \otimes \mathrm{Sym}_2\, G \to 0 \\
\vdots & & & & & \\
\mathcal{C}_\phi^{f-g-1}: & 0 \to & \bigwedge^f F \otimes \bigwedge^g G^* \otimes D_1(G^*) \xrightarrow{\eta_\phi} & \bigwedge^{f-1} F \otimes \bigwedge^g G^* \otimes D_0(G^*) \xrightarrow{\bigwedge^g \phi} \dots \xrightarrow{\mathrm{Kos}_\phi} & \bigwedge^1 F \otimes \mathrm{Sym}_{f-g-2}\, G \xrightarrow{\mathrm{Kos}_\phi} & \bigwedge^0 F \otimes \mathrm{Sym}_{f-g-1}\, G \to 0 \\
\mathcal{C}_\phi^{f-g}: & 0 \to & \bigwedge^f F \otimes \bigwedge^g G^* \otimes D_0(G^*) \xrightarrow{\bigwedge^g \phi} & \bigwedge^{f-g} F \otimes \mathrm{Sym}_0\, G \xrightarrow{\mathrm{Kos}_\phi} \dots \xrightarrow{\mathrm{Kos}_\phi} & \bigwedge^1 F \otimes \mathrm{Sym}_{f-g-1}\, G \xrightarrow{\mathrm{Kos}_\phi} & \bigwedge^0 F \otimes \mathrm{Sym}_{f-g}\, G \to 0 \\
\mathcal{C}_\phi^{f-g+1}: & 0 \to & \bigwedge^{g-1} F \otimes \mathrm{Sym}_0\, G \xrightarrow{\mathrm{Kos}_\phi} & \bigwedge^{g} F \otimes \mathrm{Sym}_1\, G \xrightarrow{\mathrm{Kos}_\phi} \dots \xrightarrow{\mathrm{Kos}_\phi} & \bigwedge^1 F \otimes \mathrm{Sym}_{f-g}\, G \xrightarrow{\mathrm{Kos}_\phi} & \bigwedge^0 F \otimes \mathrm{Sym}_{f-g+1}\, G \to 0 \\
\vdots & & & & &
\end{array}
$$

In particular, if $f = g + 1$, then

$$
\begin{array}{llllll}
 & \text{position } 3 & \text{position } 2 & \text{position } 1 & \text{position } 0 & \text{position } -1 \\
\vdots & & & & & \\
\mathcal{C}^{-2}: & & 0 \to \bigwedge^f F \otimes \bigwedge^g G^* \otimes D_3 G^* & \to \bigwedge^{f-1} F \otimes \bigwedge^g G^* \otimes D_2 G^* & \to \bigwedge^{f-2} F \otimes \bigwedge^g G^* \otimes D_1 G^* & \to \bigwedge^{f-3} F \otimes \bigwedge^g G^* \otimes D_0 G^* \quad \to 0 \\
\mathcal{C}^{-1}: & & 0 \to \bigwedge^f F \otimes \bigwedge^g G^* \otimes D_2 G^* & \to \bigwedge^{f-1} F \otimes \bigwedge^g G^* \otimes D_1 G^* & \to \bigwedge^{f-2} F \otimes \bigwedge^g G^* \otimes D_0 G^* & \to 0 \\
\mathcal{C}^{0}: & & 0 \to \underbrace{\bigwedge^f F \otimes \bigwedge^g G^* \otimes D_1 G^*}_{\cong G^*} & \to \underbrace{\bigwedge^g F \otimes \bigwedge^g G^*}_{\cong F^*} & \to \underbrace{\bigwedge^0 F \otimes \mathrm{Sym}_0\, G}_{=R} & \to 0 \\
\mathcal{C}^{1}: & & 0 \to \underbrace{\bigwedge^f F \otimes \bigwedge^g G^* \otimes D_0 G^*}_{\cong R} & \to \underbrace{\bigwedge^1 F \otimes \mathrm{Sym}_0\, G}_{=F} & \to \underbrace{\bigwedge^0 F \otimes \mathrm{Sym}_1\, G}_{=G} & \to 0 \\
\mathcal{C}^{2}: & & 0 \to \bigwedge^2 F \otimes \mathrm{Sym}_0\, G & \to \bigwedge^1 F \otimes \mathrm{Sym}_1\, G & \to \bigwedge^0 F \otimes \mathrm{Sym}_2\, G & \to 0 \\
\mathcal{C}^{3}: & 0 \to \bigwedge^3 F \otimes \mathrm{Sym}_0\, G & \to \bigwedge^2 F \otimes \mathrm{Sym}_1\, G & \to \bigwedge^1 F \otimes \mathrm{Sym}_2\, G & \to \bigwedge^0 F \otimes \mathrm{Sym}_3\, G & \to 0 \\
\vdots & & & & &
\end{array}
$$

Do notice that when $f = g + 1$, then $\mathcal{C}^0$ has a chance of being the complex in the Hilbert-Burch Theorem (It is.); and $\mathcal{C}^1$ has a chance of being the dual of the complex in the Hilbert-Burch Theorem (It is.).

4.A. **A brief discussion of the Divisor Class Group.** Divisor Class Group makes sense for a Krull Domain.

In this discussion, let A be a Noetherian domain and $\operatorname{ff}(A)$ be the fraction field of A. Then A is a Krull domain if and only if A is normal. The domain A is normal if A is integrally closed in its fraction field. (That is, if $f(x)$ is a monic polynomial in $A[x]$ and α is in the fraction field of A with $f(\alpha)=0$, then $\alpha \in A$.)

A finitely generated A-submodule of $\operatorname{ff}(A)$ is called a fractional ideal of A. A fractional ideal I of A is divisorial if $(I^{-1})^{-1}$, where $I^{-1} = A :_{\operatorname{ff}(A)} I$. One can notice that

$$I^{-1} \cong \operatorname{Hom}_A(I, A).$$

If I is a fractional ideal of A, then $(I^{-1})^{-1}$ is divisorial. Every principal ideal is divisorial. A Noetherian domain is a UFD if and only if every divisorial ideal is principal.

For example,

$$\frac{\boldsymbol{k}[x,y,u,v]}{(xu-yv)}$$

is not a UFD. The ideal (u,v) is divisorial but not principal. I think $(u,v)^{-1}$ is $(1,\frac{x}{v})$. (You might want to check that claim and fix it, if necessary.) Similarly, $\mathbb{Z}[\sqrt{-5}]$ is not a UFD. The ideal $(2, 1+\sqrt{-5})$ is divisorial but not principal. I think $(2,1+\sqrt{-5})^{-1}$ is $(1, \frac{1-\sqrt{-5}}{2})$. (Again, you might want to check this claim, and fix it if necessary.) In each of these examples the inverse is isomorphic to the original ideal (because the DCG is $\mathbb{Z}/(2)$).

If A is a Noetherian normal domain, then the set of divisorial ideals forms a group under I times J equals $((IJ)^{-1})^{-1}$ and $^{-1}$ is the inverse operation. The divisor class group of A is the group of divisorial ideals modded out by the group of principal ideals.

Go back to the usual setup of Section 4, in the generic case. In other words, let $\phi : F \to G$ be a homomorphism of free $R = R_0[\{x_{i,j}\}]$ modules with $\operatorname{rank} G = g \le f = \operatorname{rank} F$ and ϕ equal to the matrix with variable entries $\phi = (x_{i,j})$. Let $A = R/I_g(\phi)$. Then the divisor class group of A is equal to $\mathcal{C}\ell(R_0) \oplus \mathbb{Z}$ and the summand $\mathbb{Z}$ is generated by I with $I \cong \mathrm{H}_0(\mathcal{C}^1)$. Furthermore, $\mathrm{H}_0(\mathcal{C}^q)$ is isomorphic to the divisorial ideal I^q for $-1 \le q$. Furthermore, (if R_0 is Cohen-Macaulay, then) I^q is a Cohen-Macaulay A-module if and only if $-1 \le q \le f-g+1$. One direction of the most recent claim is easy to see (because we see that an R-module M is Cohen-Macaulay by seeing that $\operatorname{grade} \operatorname{ann}_R M = \operatorname{pd}_R M$ (since R is Cohen-Macaulay and M has a homogeneous resolution.)) But I am also asserting that $[I]^q$ is not Cohen-Macaulay for $q \le -2$ (and we do not know the free resolution of such $[I]^q$). I used $[I]$ to mean the class of I in the Divisor Class Group.

4.B. **The Generalized Eagon-Northcott complexes include the complexes from the Hilbert-Burch Theorem when ϕ is almost square.** Recall the generalized Eagon-Northcott complexes:

$$
\begin{array}{rlrrrr}
 & & \text{position } f-g+1 & \text{position } f-g & \text{position } 1 & \text{position } 0 \\
\vdots & & & & & \\
\mathcal{C}^{-1}_\phi : & 0 \to & \bigwedge^f F \otimes \bigwedge^g G^* \otimes D_{f-g+1}(G^*) \xrightarrow{\eta_\phi} & \bigwedge^{f-1} F \otimes \bigwedge^g G^* \otimes D_{f-g}(G^*) \xrightarrow{\eta_\phi} \dots \xrightarrow{\eta_\phi} & \bigwedge^g F \otimes \bigwedge^g G^* \otimes D_1(G^*) \xrightarrow{\eta_\phi} & \bigwedge^{g-1} F \otimes \bigwedge^g G^* \otimes D_0(G^*) \to 0 \\
\mathcal{C}^{0}_\phi : & 0 \to & \bigwedge^f F \otimes \bigwedge^g G^* \otimes D_{f-g}(G^*) \xrightarrow{\eta_\phi} & \bigwedge^{f-1} F \otimes \bigwedge^g G^* \otimes D_{f-g-1}(G^*) \xrightarrow{\eta_\phi} \dots \xrightarrow{\eta_\phi} & \bigwedge^g F \otimes \bigwedge^g G^* \otimes D_0(G^*) \xrightarrow{\bigwedge^g \phi} & \bigwedge^0 F \otimes \operatorname{Sym}_0 G \to 0 \\
\mathcal{C}^{1}_\phi : & 0 \to & \bigwedge^f F \otimes \bigwedge^g G^* \otimes D_{f-g-1}(G^*) \xrightarrow{\eta_\phi} & \bigwedge^{f-1} F \otimes \bigwedge^g G^* \otimes D_{f-g-2}(G^*) \xrightarrow{\eta_\phi} \dots \xrightarrow{\bigwedge^g \phi} & \bigwedge^1 F \otimes \operatorname{Sym}_0 G \xrightarrow{\mathrm{Kos}_\phi} & \bigwedge^0 F \otimes \operatorname{Sym}_1 G \to 0 \\
\mathcal{C}^{2}_\phi : & 0 \to & \bigwedge^f F \otimes \bigwedge^g G^* \otimes D_{f-g-2}(G^*) \xrightarrow{\eta_\phi} & \bigwedge^{f-1} F \otimes \bigwedge^g G^* \otimes D_{f-g-3}(G^*) \xrightarrow{\eta_\phi} \dots \xrightarrow{\mathrm{Kos}_\phi} & \bigwedge^1 F \otimes \operatorname{Sym}_1 G \xrightarrow{\mathrm{Kos}_\phi} & \bigwedge^0 F \otimes \operatorname{Sym}_2 G \to 0 \\
\vdots & & & & &
\end{array}
$$

When $f = g + 1$, then the generalized Eagon-Northcott complexes become:

$$
\begin{array}{lllllll}
 & \text{position } 3 & \text{position } 2 & \text{position } 1 & \text{position } 0 & \text{position } -1 & \\
\vdots & & & & & & \\
\mathcal{C}^{-2} : & & 0 \to \bigwedge^f F \otimes \bigwedge^g G^* \otimes D_3 G^* & \to \bigwedge^{f-1} F \otimes \bigwedge^g G^* \otimes D_2 G^* & \to \bigwedge^{f-2} F \otimes \bigwedge^g G^* \otimes D_1 G^* & \to \bigwedge^{f-3} F \otimes \bigwedge^g G^* \otimes D_0 G^* & \to 0 \\
\mathcal{C}^{-1} : & & 0 \to \bigwedge^f F \otimes \bigwedge^g G^* \otimes D_2 G^* & \to \bigwedge^{f-1} F \otimes \bigwedge^g G^* \otimes D_1 G^* & \to \bigwedge^{f-2} F \otimes \bigwedge^g G^* \otimes D_0 G^* & \to 0 & \\
\mathcal{C}^{0} : & & 0 \to \underbrace{\bigwedge^f F \otimes \bigwedge^g G^* \otimes D_1 G^*}_{\cong G^*} & \to \underbrace{\bigwedge^g F \otimes \bigwedge^g G^*}_{\cong F^*} & \to \underbrace{\bigwedge^0 F \otimes \operatorname{Sym}_0 G}_{= R} & \to 0 & \\
\mathcal{C}^{1} : & & 0 \to \underbrace{\bigwedge^f F \otimes \bigwedge^g G^* \otimes D_0 G^*}_{\cong R} & \to \underbrace{\bigwedge^1 F \otimes \operatorname{Sym}_0 G}_{= F} & \to \underbrace{\bigwedge^0 F \otimes \operatorname{Sym}_1 G}_{= G} & \to 0 & \\
\mathcal{C}^{2} : & & 0 \to \bigwedge^2 F \otimes \operatorname{Sym}_0 G & \to \bigwedge^1 F \otimes \operatorname{Sym}_1 G & \to \bigwedge^0 F \otimes \operatorname{Sym}_2 G & \to 0 & \\
\mathcal{C}^{3} : & 0 \to \bigwedge^3 F \otimes \operatorname{Sym}_0 G & \to \bigwedge^2 F \otimes \operatorname{Sym}_1 G & \to \bigwedge^1 F \otimes \operatorname{Sym}_2 G & \to \bigwedge^0 F \otimes \operatorname{Sym}_3 G & \to 0 & \\
\vdots & & & & & &
\end{array}
$$

Do notice that when $f = g + 1$, then $\mathcal{C}^0$ has a chance of being the complex in the Hilbert-Burch Theorem (It is.); and $\mathcal{C}^1$ has a chance of being the dual of the complex in the Hilbert-Burch Theorem (It is.).

4.C. **Symmetric Algebras, Exterior Algebras, and Divided Power modules.**

Definition 4.4. Let R be a commutative ring and M be an R-module. The symmetric algebra of M is a commutative R-algebra $\operatorname{Sym}(M)$ together with and R-module homomorphism $M \xrightarrow{i} \operatorname{Sym}(M)$ which satisfies the following universal mapping property: Whenever S is a commutative R-algebra and $M \xrightarrow{f} S$ is an R-module homomorphism, then there exits a unique R-algebra homomorphism $\tilde{f}$ such that

$$\begin{tikzcd}
\operatorname{Sym}(M) \arrow[dr, dashed, "\exists!\tilde{f}"] & \\
M \arrow[u, "i"] \arrow[r, "f"'] & S
\end{tikzcd}$$

commutes.

Remarks. The symmetric algebra always exists. It is the tensor algebra

$$T(M) = R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \dots$$

modded out by the two-sided ideal generated by $\{m \otimes m' - m' \otimes m \mid m, m' \in M\}$.

Notice that $\operatorname{Sym}(M)$ is automatically a graded R-module; because $T(M)$ is a graded R-algebra and one is modding out by a homogeneous ideal.

In practice, $\operatorname{Sym}(M)$ is easy to deal with. If R is Noetherian, M is finitely generated, and

$$R^s \xrightarrow{A} R^t \to M \to 0$$

is exact, then

$$\operatorname{Sym}(M) = R[T_1, \dots, T_t]/([T_1, \dots, T_t]A).$$

Of course, if M is a free module of rank t, then $\operatorname{Sym}(M)$ is the polynomial ring in t variables with coefficients from R and $\operatorname{Sym}_q(M)$ is the free R-module on the monomials of degree q in t variables.

Definition 4.5. Let R be a commutative ring and M be an R-module. The exterior algebra of M is an associative R-algebra $\bigwedge^{\bullet}(M)$ together with and R-module homomorphism $M \xrightarrow{i} \bigwedge^{\bullet}(M)$ which satisfies the following universal mapping property: Whenever A is an associative R-algebra and $M \xrightarrow{f} A$ is an R-module homomorphism with the property that $(f(m))^2 = 0$ for all $m \in M$, then there exits a unique R-algebra homomorphism $\tilde{f}$ such that

$$\begin{tikzcd}
\bigwedge^{\bullet}(M) \arrow[dr, dashed, "\exists!\tilde{f}"] & \\
M \arrow[u, "i"] \arrow[r, "f"'] & A
\end{tikzcd}$$

commutes.

Remarks. The exterior algebra always exists. It is the tensor algebra

$$T(M) = R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \dots$$

modded out by the two-sided ideal generated by $\{m \otimes m \mid m \in M\}$.

Notice that $\bigwedge^\bullet(M)$ is automatically a graded R-module; because $T(M)$ is a graded R-algebra and one is modding out by a homogeneous ideal.

4.6. If F is a finitely generated free R-module, then

$$\bigwedge^\bullet F \text{ is a } \bigwedge^\bullet F^* - \text{module} \quad \text{and} \quad \bigwedge^\bullet F^* \text{ is a } \bigwedge^\bullet F - \text{module}.$$

Both actions are induced by evaluation

$$v_1(w_1) = w_1(v_1) \in R,$$

for $v_1 \in F$ and $w_1 \in F^*$. Furthermore, all module actions and multiplications are graded-commutative and associative:

$$v_1(w_1 \wedge w_1' \wedge w_1'') = v_1(w_1) \cdot w_1' \wedge w_1'' - v_1(w_1') \cdot w_1 \wedge w_1'' + v_1(w_1'') \cdot w_1 \wedge w_1'$$

and

$$(v_1 \wedge v_1' \wedge v_1'')(w_r) = v_1(v_1'(v_1''(w_r))).$$

In particular,

$$(v_1 \wedge v_1')(w_1 \wedge w_1') = v_1'(w_1) \cdot v_1(w_1') - v_1(w_1) \cdot v_1'(w_1') = -\begin{vmatrix} v_1(w_1) & v_1'(w_1) \\ v_1(w_1') & v_1'(w_1') \end{vmatrix}.$$

Let $\phi : F \to G$ be a homomorphism of free R-modules, $a_1, \dots, a_f$ be a basis for F and $b_1, \dots, b_g$ be a basis for G with dual basis $b_1^*, \dots, b_g^*$ then

$$\bigwedge^2 F \otimes \bigwedge^2 G^* \xrightarrow{\bigwedge^2 \phi \otimes 1} \bigwedge^2 G \otimes \bigwedge^2 G^* \xrightarrow{\text{module action}} R$$

sends $(a_{c_1} \wedge a_{c_2}) \otimes (b_{r_1}^* \wedge b_{r_2}^*)$ to

$$[\phi(a_{c_1}) \wedge \phi(a_{c_2})](b_{r_1}^* \wedge b_{r_2}^*) = -\begin{vmatrix} [\phi(a_{c_1})](b_{r_1}^*) & [\phi(a_{c_2})](b_{r_1}^*) \\ [\phi(a_{c_1})](b_{r_2}^*) & [\phi(a_{c_2})](b_{r_2}^*) \end{vmatrix} = -\begin{vmatrix} \sum_{\ell=1}^{f} \phi_{\ell,c_1} b_\ell(b_{r_1}^*) & \sum_{\ell=1}^{f} \phi_{\ell,c_2} b_\ell(b_{r_1}^*) \\ \sum_{\ell=1}^{f} \phi_{\ell,c_1} b_\ell(b_{r_2}^*) & \sum_{\ell=1}^{f} \phi_{\ell,c_2} b_\ell(b_{r_2}^*) \end{vmatrix}$$

$$= -\begin{vmatrix} \phi_{r_1,c_1} & \phi_{r_1,c_2} \\ \phi_{r_2,c_1} & \phi_{r_2,c_2} \end{vmatrix},$$

where $\phi(a_c) = \sum_{\ell=1}^{g} \phi_{\ell,c} b_\ell$.

4.7. Now that we know that $\bigwedge^\bullet F$ is a $\bigwedge^\bullet F^*$-algebra, we have a very clean way to say Koszul complex. Let F be a finitely generated free R-module of rank f and σ be an element of F^*. Then the Koszul complex associated to σ is

$$0 \to \bigwedge^f F \stackrel{\sigma}{\to} \bigwedge^{f-1} F \stackrel{\sigma}{\to} \cdots \stackrel{\sigma}{\to} \bigwedge^1 F \stackrel{\sigma}{\to} \bigwedge^0 F \to 0.$$

4.8. Two comments added before the lecture on March 20, 2018.

(a) Last time I used the action of $\bigwedge^\bullet F^*$ on $\bigwedge^\bullet F$ to give a coordinate-free description of a Koszul complex; **BUT** I forgot to point out **why** the construction yields a complex. If F is a free R-module of rank n and $\sigma \in F^*$, then the Koszul complex associated to σ is

$$0 \to \bigwedge^f F \stackrel{\sigma}{\to} \bigwedge^{f-1} F \stackrel{\sigma}{\to} \cdots \stackrel{\sigma}{\to} \bigwedge^1 F \stackrel{\sigma}{\to} \bigwedge^0 F \to 0.$$

This collection of homomorphisms is a complex because

$$\begin{aligned} \sigma(\sigma(\theta_r)) &= (\sigma \wedge \sigma)(\theta_r) && \text{because } \textstyle\bigwedge^\bullet F \text{ is a } \textstyle\bigwedge^\bullet F^* \text{ module} \\ &= 0 && \text{because } \sigma \in \textstyle\bigwedge^1 F^* \text{ and } \textstyle\bigwedge^* F^* \text{ is an alternating graded algebra.} \end{aligned}$$

(b) At some point I used the phrase “anti-canonical module”. I did not define it and what I said about it was mainly wrong. I can do better. Let R be a normal domain; so R has a divisor class group. The canonical module of R is usually an ideal of R. (Recall that if $R = P/I$ where P is a polynomial ring over a field and I is a homogeneous ideal, and R is a perfect R-module (that is $\operatorname{pd}_P R \le \operatorname{grade} I$), then the canonical module of R is $\operatorname{Ext}^{\operatorname{pd}_P R}(R, P)$.) Indeed, the canonical module of R is usually a divisorial ideal of R. Anyhow, if the canonical module of R is a divisorial ideal ω of R, then the anticanonical module of R is ω^{-1}.

In the context of the generic Eagon-Northcott complexes, then $\mathcal{C}^{f-g}$ resolves ω. If J is $\mathrm{H}_0(\mathcal{C})$, then we know the resolution of $q[J]$ for $-1 \le q$. We do not know the resolution of $q[J]$ for $q < -2$. In particular, we know the resolution of the anticanonical module only when $f = g + 1$. In this case $\mathcal{C}^{-1}$ resolves $-[J]$. (This is exactly what I heard Jesse say. I generalized his statement recklessly.)

4.9. If G is a finitely generated free R-module, then $D_\bullet G$ is the graded dual of the R-module $\operatorname{Sym}_\bullet G$. In other words,

$$D_\bullet G = \bigoplus_{i=0}^{\infty} D_i G^*$$

and $D_i G^* = \operatorname{Hom}_R(\operatorname{Sym}_i G, R)$. The R-module $D_\bullet G$ is a $\operatorname{Sym}_\bullet G$-module under the action $\operatorname{poly}_i \in \operatorname{Sym}_i G$ sends $w_j \in D_j G^*$ to the element $\operatorname{poly}_i(w_j)$ in $D_{j-i} G^*$, where $\operatorname{poly}_i(w_j)$ sends $\operatorname{poly}_{j-i}$ to $(\operatorname{poly}_{j-i} \operatorname{poly}_i)(w_j)$. If $x_1, \dots, x_g$ is basis for G, then

$$\binom{x_1, \dots, x_g}{i} = \text{ the set of monomials of degree } i \text{ in } x_1, \dots, x_g$$

is a basis for $\operatorname{Sym}_i G$ and $\{m^* \mid m \in \binom{x_1,\dots,x_g}{i}\}$ is a basis for $D_i G^*$. Observe that

$$x_\ell(m^*) = \begin{cases} 0 & \text{if } x_\ell \not| m \\ (\frac{m}{x_\ell})^* & \text{if } x_\ell | m \end{cases}$$

4.D. **Every homomorphism of finitely generated free R-modules gives rise to a family of Koszul complexes.** Let $\phi : F \to G$ be a homomorphism of finitely generated free R-modules. Let $\widetilde{\phi} : F \otimes_R \operatorname{Sym}_\bullet^R G \to \operatorname{Sym}_\bullet^R G$ be the composition

$$F \otimes_R \operatorname{Sym}_\bullet^R G \xrightarrow{\phi \otimes 1} G \otimes_R \operatorname{Sym}_\bullet^R G \xrightarrow{\text{mult}} \operatorname{Sym}_\bullet^R G.$$

Notice that $F \otimes_R \operatorname{Sym}_\bullet^R G$ is a free $\operatorname{Sym}_\bullet^R G$-module of rank f. This is the usual set-up to make the Koszul complex

$$(\textstyle\bigwedge^\bullet_{\operatorname{Sym}_\bullet^R G}(F \otimes_R \operatorname{Sym}_\bullet^R G), \widetilde{\phi}) :$$

$$(\dagger) \quad 0 \to \textstyle\bigwedge^f_{\operatorname{Sym}_\bullet^R G}(F \otimes_R \operatorname{Sym}_\bullet^R G) \xrightarrow{\widetilde{\phi}} \dots \xrightarrow{\widetilde{\phi}} \bigwedge^1_{\operatorname{Sym}_\bullet^R G}(F \otimes_R \operatorname{Sym}_\bullet^R G) \xrightarrow{\widetilde{\phi}} \bigwedge^0_{\operatorname{Sym}_\bullet^R G}(F \otimes_R \operatorname{Sym}_\bullet^R G) \to 0.$$

Notice that

$$\textstyle\bigwedge^\bullet_{\operatorname{Sym}_\bullet^R G}(F \otimes_R \operatorname{Sym}_\bullet^R G) \cong \bigwedge^\bullet_R F \otimes_R \operatorname{Sym}_\bullet^R G;$$

and therefore, (†) decomposes into a direct sum of complexes of finitely generated free R-modules:

$$\begin{tikzcd}
 & & 0 \arrow[r] & \bigwedge^0 F \otimes \operatorname{Sym}_0 G \arrow[r] & 0 \\
 & 0 \arrow[r] & \bigwedge^1 F \otimes \operatorname{Sym}_0 G \arrow[r, "\widetilde{\phi}"] & \bigwedge^0 F \otimes \operatorname{Sym}_1 G \arrow[r] & 0 \\
0 \arrow[r] & \bigwedge^2 F \otimes \operatorname{Sym}_0 G \arrow[r, "\widetilde{\phi}"] & \bigwedge^1 F \otimes \operatorname{Sym}_1 G \arrow[r, "\widetilde{\phi}"] & \bigwedge^0 F \otimes \operatorname{Sym}_2 G \arrow[r] & 0 \\
 & & \vdots & &
\end{tikzcd}$$

These maps $\widetilde{\phi}$ are very understandable:

$$\widetilde{\phi} : \textstyle\bigwedge^i F \otimes \operatorname{Sym}_j G \to \bigwedge^{i-1} F \otimes \operatorname{Sym}_{j+1} G$$

is

$$\widetilde{\phi}(a_1 \wedge \dots \wedge a_i \otimes \operatorname{poly}_j) = \sum_{\ell=1}^{i} (-1)^{\ell+1} (a_1 \wedge \dots \wedge \widehat{a_\ell} \wedge \dots \wedge a_i) \otimes \phi(a_i) \operatorname{poly}_j .$$

4.E. **The “Eagon-Northcott maps”.** The Eagon-Northcott map

$$\operatorname{EN}_\phi : \textstyle\bigwedge^i F \otimes D_j G^* \to \bigwedge^{i-1} F \otimes D_{j-1} G^*$$

is

$$\operatorname{EN}_\phi(a_1 \wedge \dots \wedge a_i \otimes w_j) = \sum_{\ell=1}^{i} (-1)^{i+1} (a_1 \wedge \dots \wedge \widehat{a_\ell} \wedge \dots \wedge a_i) \otimes [\phi(a_i)](w_j),$$

with the a’s in F and $w_j \in D_j G^*$. We can express this as

$$\operatorname{EN}_\phi(\theta_i \otimes w_j) = \sum_{\ell=1}^{f} x_i^*(\theta_i) \otimes [\phi(x_i)](w_j)$$

where $\theta_i \in \bigwedge^i F$, and $x_1, \dots, x_f$ and $x_1^*, \dots, x_f^*$ is a pair of dual bases for F and F^*, respectively.

The element $\sum_\ell x_\ell \otimes x_\ell^* \in F \otimes F^*$ is canonical; that is, it is independent of bases! Indeed, the evaluation map

$$\operatorname{ev} : F^* \otimes F \to R$$

is canonical; the dual

$$\operatorname{ev}^* : R \to F \otimes F^*$$

is canonical; the element 1 in R is canonical; and

$$\mathrm{ev}^*(1) = \sum_\ell x_\ell \otimes x_\ell^*$$

is canonical.

Claim.

The complexes

$$\cdots \to \bigwedge^i F \otimes D_j G^* \otimes \bigwedge^f F^* \xrightarrow{\mathrm{EN}_\phi} \bigwedge^{i-1} F \otimes D_{j-1} G^* \otimes \bigwedge^f F^* \to \cdots$$

and

$$\cdots \to \mathrm{Hom}_R(\bigwedge^{f-i} F \otimes \mathrm{Sym}_j G, R) \xrightarrow{\mathrm{Kos}_\phi^*} \mathrm{Hom}_R(\bigwedge^{f-i+1} F \otimes \mathrm{Sym}_{j-1} G, R) \to \cdots$$

are canonically isomorphic (up to sign). Define

$$\langle -, - \rangle : \left(\bigwedge^i F \otimes D_j G^* \otimes \bigwedge^f F^*\right) \otimes \left(\bigwedge^{f-i} F \otimes \mathrm{Sym}_j G\right) \to R$$

by

$$\langle (\theta_i \otimes w_j \otimes \Theta_f) \otimes (\theta_{f-i} \otimes \mathrm{poly}_j) \rangle = (\theta_i \wedge \theta_{f-i})(\Theta_f) \cdot w_j(\mathrm{poly}_j).$$

It suffices to show that

$$\langle (\theta_i \otimes w_j \otimes \Theta_f) \otimes \mathrm{Kos}_\phi(\theta_{f-i+1} \otimes \mathrm{poly}_{j-1}) \rangle = \langle (\mathrm{EN}_\phi(\theta_i \otimes w_j) \otimes \Theta_f) \otimes (\theta_{f-i+1} \otimes \mathrm{poly}_{j-1}) \rangle$$

(up to sign). The left side is

$$\begin{aligned} &\langle (\theta_i \otimes w_j \otimes \Theta_f) \otimes \mathrm{Kos}_\phi(\theta_{f-i+1} \otimes \mathrm{poly}_{j-1}) \rangle \\ =& \sum_\ell \langle (\theta_i \otimes w_j \otimes \Theta_f) \otimes [x_\ell^*(\theta_{f-i+1}) \otimes [\phi(x_\ell)] \cdot \mathrm{poly}_{j-1}] \rangle \\ =& \sum_\ell [\theta_i \wedge x_\ell^*(\theta_{f-i+1})](\Theta_f) \cdot w_j([\phi(x_\ell)] \cdot \mathrm{poly}_{j-1}) \end{aligned}$$

Keep in mind that $\theta_i \wedge \theta_{f-i+1} \in \bigwedge^{f+1} F = 0$; so

$$0 = x_\ell^*(\theta_i \wedge \theta_{f-i+1}) = x_\ell^*(\theta_i) \wedge \theta_{f-i+1} + (-1)^i \theta_i \wedge x_\ell^*(\theta_{f-i+1}).$$

The left side is

$$(-1)^{i-1} \sum_\ell [x_\ell^*(\theta_i) \wedge \theta_{f-i+1}](\Theta_f) \cdot w_j([\phi(x_\ell)](\mathrm{poly}_{j-1})).$$

The right side

$$\begin{aligned} &\langle (\mathrm{EN}_\phi(\theta_i \otimes w_j) \otimes \Theta_f) \otimes (\theta_{f-i+1} \otimes \mathrm{poly}_{j-1}) \rangle \\ =& \sum_\ell \langle (x_\ell^*(\theta_i) \otimes [\phi(x_\ell)](w_j)) \otimes \Theta_f) \otimes (\theta_{f-i+1} \otimes \mathrm{poly}_{j-1}) \rangle \\ =& \sum_\ell [(x_\ell^*(\theta_i) \wedge \theta_{f-i+1}](\Theta_f) \cdot \big([\phi(x_\ell)](w_j)\big)(\mathrm{poly}_{j-1}). \end{aligned}$$

The left side and the right side differ by a factor of $(-1)^{i-1}$.

4.F. **Co-multiplication in the exterior algebra.** I want an algebraic framework for

$$\bigwedge^3 F \to \bigwedge^2 F \otimes \bigwedge^1 F$$

with

$$a_1 \wedge a_1' \wedge a_1'' \mapsto a_1' \wedge a_1'' \otimes a_1 - a_1 \wedge a_1'' \otimes a_1' + a_1 \wedge a_1' \otimes a_1'', \tag{4.9.1}$$

for $a_1, a_1', a_1'' \in F$.

One can view $\bigwedge F \otimes_R \bigwedge F$ as an algebra with multiplication

$$(a_i \otimes b_j) \cdot (a_k' \otimes b_\ell') = (-1)^{jk}(a_i a_k' \otimes b_j b_\ell'),$$

for $\square_r \in \bigwedge^r F$. This product makes

$$\bigwedge^\bullet (F \oplus F) \quad \text{and} \quad \bigwedge^\bullet F \otimes \bigwedge^\bullet F$$

isomorphic as algebras and it also makes multiplication

$$\bigwedge^\bullet F \otimes \bigwedge^\bullet F \to \bigwedge^\bullet F$$

be an algebra map.

The algebra map

$$\Delta : \bigwedge^\bullet F \to \bigwedge^\bullet F \otimes \bigwedge^\bullet F$$

which extends

$$\Delta(a_1) = a_1 \otimes 1 + 1 \otimes a_1$$

for $a_1 \in F$ is called <u>co-multiplication</u>. Observe that

$$\begin{aligned} \Delta(a_1 \wedge a_1') &= (a_1 \otimes 1 + 1 \otimes a_1) \cdot (a_1' \otimes 1 + 1 \otimes a_1') \\ &= a_1 a_1' \otimes 1 + \Big(a_1 \otimes a_1' - a_1' \otimes a_1\Big) + 1 \otimes a_1 \wedge a_1', \end{aligned}$$

for $a_1, a_1' \in F$. Similarly

$$\begin{aligned} \Delta(a_1 \wedge a_1' \wedge a_1'') &= \Delta(a_1 \wedge a_1') \cdot \Delta(a_1'') \\ &= \Big[a_1 \wedge a_1' \otimes 1 + \Big(a_1 \otimes a_1' - a_1' \otimes a_1\Big) + 1 \otimes a_1 \wedge a_1'\Big](a_1'' \otimes 1 + 1 \otimes a_1''), \\ &= a_1 \wedge a_1' \wedge a_1'' \otimes 1 + \begin{pmatrix} a_1 \wedge a_1' \otimes a_1'' \\ -a_1 \wedge a_1'' \otimes a_1' \\ +a_1' \wedge a_1'' \otimes a_1 \end{pmatrix} + \begin{pmatrix} a_1 \otimes a_1' \wedge a_1'' \\ -a_1' \otimes a_1 \wedge a_1'' \\ +a_1'' \otimes a_1 \wedge a_1' \end{pmatrix} + 1 \otimes a_1 \wedge a_1' \wedge a_1'' \end{aligned}$$

for $a_1, a_1', a_1'' \in F$. So (4.9.1) is a component of the co-multiplication map

$$\Delta : \bigwedge^\bullet F \to \bigwedge^\bullet F \otimes \bigwedge^\bullet F.$$

The map

$$\bigwedge^{a+g} F \otimes \bigwedge^g G^* \otimes D_0 G^* \to \bigwedge^a F \otimes \operatorname{Sym}_0 G$$

in the Generalized Eagon-Northcott complexes is the composition

$$\bigwedge^{a+g} F \otimes \bigwedge^g G^* \otimes D_0 G^* = \bigwedge^{a+g} F \otimes \bigwedge^g G^* \xrightarrow{\Delta \otimes 1} \bigwedge^a F \otimes \bigwedge^g F \otimes \bigwedge^g G^*$$

$$\xrightarrow{1 \otimes \bigwedge^g \phi \otimes 1} \bigwedge^a F \otimes \bigwedge^g G \otimes \bigwedge^g G^* \xrightarrow{1 \otimes \text{module action}} \bigwedge^a F = \bigwedge^a F \otimes \operatorname{Sym}_0 G.$$

In practice this is straightforward. Start with $f_1 \wedge \ldots \wedge f_{g+a}$ with $f_i \in F$. For each choice of g subscripts write down the corresponding signed $g \times g$ minor of ϕ next to the wedge product of the remaining a f's.

Example 4.10. Take $f = g+1$. Record $\mathcal{C}^1$:

$$0 \to \bigwedge^{g+1} F \otimes \bigwedge^g G^* \otimes D_0 G^* \to \underbrace{\bigwedge^1 F \otimes \operatorname{Sym}_0 G \to \bigwedge^0 F \otimes \operatorname{Sym}_1 G}_{\phi: F \to G} \to 0$$

The left-most map sends

$$f_1 \wedge \ldots \wedge f_{g+1} \mapsto \sum_i (-1)^{i+1} \det \phi \text{ with col } i \text{ deleted} f_i.$$

There is another way to say the map

$$\bigwedge^{a+g} F \otimes \bigwedge^g G^* \otimes D_0 G^* \to \bigwedge^a F \otimes \operatorname{Sym}_0 G$$

in the Generalized Eagon-Northcott complexes:

$$\theta_{a+g} \otimes w_g \mapsto \Big([\bigwedge^g \phi^*](w_g) \Big)(\theta_{a+g}).$$

4.G. **At this point we know all of the maps in the $\mathcal{C}$. Lets make sure that they form complexes.** It suffices to look at

$$\bigwedge^{g+a} F \otimes \bigwedge^g G^* \otimes D_0 G^* \to \bigwedge^a F \otimes \operatorname{Sym}_0 G \to \bigwedge^{a-1} F \otimes \operatorname{Sym}_1 G \tag{4.10.1}$$

and

$$\bigwedge^{g+a+1} F \otimes \bigwedge^g G^* \otimes D_1 G^* \to \bigwedge^{g+a} F \otimes \bigwedge^g G^* \otimes D_0 G^* \to \bigwedge^a F \otimes \operatorname{Sym}_0 G \tag{4.10.2}$$

Fix $\theta_{g+a} \in \bigwedge^{g+a} F$ and $w_g \in \bigwedge^g G^*$. The composition (4.10.1) sends

$$\theta_{g+a} \otimes w_g \mapsto [(\bigwedge^g \phi^*)(w_g)](\theta_{g+a}) \mapsto \underbrace{\textstyle\sum_i e_i^* \Big([(\bigwedge^g \phi^*)(w_g)](\theta_{g+a}) \Big) \otimes \phi(e_i)}_{\dagger},$$

where $e_1, \ldots, e_f$ and $e_1^*, \ldots, e_f^*$ are a pair of dual bases for F and F^* respectively. (As always $\sum_i e_i \otimes e_i^*$ is a canonical element of $F \otimes F^*$.) We show that $(1 \otimes w_1)(\dagger) = 0$. Indeed,

$$\begin{aligned}
(1 \otimes w_1)(\dagger) &= \phi^*(w_1)\Big([(\bigwedge^g \phi^*)(w_g)](\theta_{g+a}) \Big) & \\
&= \Big(\phi^*(w_1) \wedge [(\bigwedge^g \phi^*)(w_g)] \Big)(\theta_{g+a}) & \text{module action!!} \\
&= \Big([(\bigwedge^{g+1} \phi^*)(w_1 \wedge w_g)] \Big)(\theta_{g+a}) & \text{the definition of } \bigwedge^\bullet \phi \\
&= 0 & \operatorname{rank} G = g.
\end{aligned}$$

Fix $\theta_{g+a+1} \in \bigwedge^{g+a} F$, $w_1 \in G^*$, and $w_g \in \bigwedge^g G^*$. The composition (4.10.2) sends

$$\begin{aligned}
\theta_{g+a+1} \otimes w_g \otimes w_1 \mapsto \textstyle\sum_i e_i^*(\theta_{g+a+1}) \otimes w_g \otimes [\phi(e_i)](w_1) \mapsto & \textstyle\sum_i [\phi(e_i)](w_1) \cdot [(\bigwedge^g \phi^*)(\omega_g)][e_i^*(\theta_{g+a+1})] \\
&= [(\bigwedge^g \phi^*)(\omega_g)]\big[[\phi^*(w_1)](\theta_{g+a+1})\big] \\
&= \big[[(\bigwedge^g \phi^*)(\omega_g)] \wedge [\phi^*(w_1)]\big](\theta_{g+a+1}) \\
&= \big[(\bigwedge^{g+1} \phi^*)(\omega_g \wedge w_1)\big](\theta_{g+a+1}) = 0
\end{aligned}$$

4.H. **The acyclicity lemma.** I think I told you that at the beginning of my career, the Theorem that I automatically used to prove that a complex is acyclic was WMACE; however, now-a-days I use the acyclicity lemma; which can be thought of as a consequence of WMACE. We will use the acyclicity lemma to prove that the generalized Eagon-Northcott complexes are acyclic.

Lemma 4.11. *Let R be a Noetherian ring and*

$$F: \quad 0 \to F_n \xrightarrow{f_n} \dots \xrightarrow{f_1} F_0 \to 0$$

be a complex of finitely generated free R-modules. If $F \otimes_R R_P$ is acyclic for all prime ideals P of R with $\operatorname{depth} R_P < n$, then F is acyclic.

Before starting the proof I want to re-write the linear algebra hypothesis of WMACE. Define

$$\begin{aligned} r_n &= \operatorname{rank} F_n, \\ r_{n-1} &= \operatorname{rank} F_{n-1} - \operatorname{rank} F_n, \\ r_{n-2} &= \operatorname{rank} F_{n-2} - \operatorname{rank} F_{n-1} + \operatorname{rank} F_n, \\ &\vdots \\ r_1 &= \operatorname{rank} F_1 - \operatorname{rank} F_2 + \operatorname{rank} F_3 - \cdots + (-1)^{n+1} \operatorname{rank} F_n. \end{aligned}$$

Observe that

$$F_k = \operatorname{rank} f_{k+1} + \operatorname{rank} f_k \text{ for } 1 \le k \le n \iff \operatorname{rank} f_k = r_k \text{ for } 1 \le k \le n.$$

If one calls r_k the expected rank of f_k, then WMACE says F is acyclic if and only if each f_k has the expected rank and $I_{r_k}(f_k)$ has grade at least k or is equal to R for $1 \le k \le n$.

The proof of the acyclicity lemma. Fix an index k with $1 \le k \le n$. Take $P \in \operatorname{Spec} R$ with $\operatorname{depth} R_P < k$. WMACE guarantees that $k \le \operatorname{grade} I_{r_k}(f_k)_P$ or $I_{r_k}(f_k)_P = R_P$. The first option is not possible; so the second option holds. Thus, $I_{r_k}(f_k)$ is not contained in P. We conclude first that $I_{r_k}(f_k)$ is not zero. (Hence f_k has the expected rank.) Secondly, if $I_{r_k}(f_k)$ is a proper ideal, then $I_{r_k}(f_k)$ is contained in some ideal Q with $\operatorname{grade} I_{r_k}(f_k) = \operatorname{depth} R_Q$[2] But $I_{r_k}(f_k)$ does not sit inside any ideal P with $\operatorname{depth} R_P < k$; thus $k \le \operatorname{grade} I_{r_k}(f_k)$. □

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[2] Let J be an ideal, $\ell = \operatorname{grade} J$, $\underline{x} = x_1, \dots, x_\ell$ be a maximal regular sequence in J. Observe that J is contained in some P which is in $\operatorname{Ass}(R/(\underline{x}))$. Observe ℓ, $\operatorname{grade} P$, and $\operatorname{depth} R_P$ all are equal. We use that

$$P \in \operatorname{Ass}(R/(\underline{x})) \implies PR_P \in \operatorname{Ass}(R/(\underline{x}))_P.$$

4.I. **Assume $f-g+1 \le \operatorname{grade} I_g(\phi)$. The acyclicity Lemma tells us how to prove that $\mathcal{C}^q$ is acyclic for $-1 \le q \le f-g+1$.** Use the acyclicity lemma. It suffices to prove that

$$\operatorname{grade} P < f-g+1 \implies (\mathcal{C}^q)_P \text{ is acyclic.}$$

But

$$\begin{aligned}\operatorname{grade} P < f-g+1 \le \operatorname{grade} I_g(\phi) &\implies [I_g(\phi)]_P = R_P\\ &\implies \text{some } g \times g \text{ minor of } \phi_P \text{ is a unit}\\ &\implies \phi_P \text{ is surjective}\end{aligned}$$

4.12. Assume $f-g+1 \le \operatorname{grade} I_g(\phi)$. To prove that $\mathcal{C}^q$ is acyclic for $-1 \le q \le f-g+1$ it suffices to prove that $\mathcal{C}^q_\phi$ is split exact for $-1 \le q \le f-g+1$ whenever R is a local ring and ϕ is surjective.

4.J. **Assume that R is local and $\phi : F \to G$ is surjective.** Let $K = \ker \phi : F \to G$. The fact that ϕ is surjective and G is free forces K to be a direct summand of F. Let $i : K \to F$ be the inclusion map and $\pi : F \to K$ be the corresponding projection. The diagram

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[dr, "1"] & F \arrow[d, "\pi"] \\
& K
\end{tikzcd}$$

commutes. We prove that the complexes

(a)

$$0 \to \bigwedge^j K \otimes \operatorname{Sym}_0 G \xrightarrow{\bigwedge^j i} \bigwedge^j F \otimes \operatorname{Sym}_0 G \xrightarrow{\mathrm{Kos}_\phi} \cdots \xrightarrow{\mathrm{Kos}_\phi} \bigwedge^0 F \otimes \operatorname{Sym}_j G \to 0,$$

(b)

$$0 \to \bigwedge^f F \otimes \bigwedge^g G^* \otimes D_j G^* \xrightarrow{\mathrm{EN}_\phi} \cdots \xrightarrow{\mathrm{EN}_\phi} \bigwedge^{f-j} F \otimes \bigwedge^g G^* \otimes D_0 G^* \xrightarrow{\bigwedge^{f-j}\pi} \bigwedge^{f-j} K \otimes \bigwedge^g G^* \otimes D_0 G^* \to 0,$$

and

(c) $\mathcal{C}^q$

are split exact for all integers j and q.

Once we prove (a), then (b) is also true by duality, and (c) is immediate. Indeed, the complex $\mathcal{C}^q$ is obtained by patching together a complex from (a) and a complex from (b). We only need worry about the patch:

$$\begin{tikzcd}
{} \arrow[r, "\mathrm{EN}"] & \bigwedge^{g+a} F \otimes \bigwedge^g G^* \otimes D_0 G \arrow[rrr, "\bigwedge^g \phi"] \arrow[dr, "\bigwedge^a \pi"'] & & & \bigwedge^a F \otimes \operatorname{Sym}_0 G \arrow[r, "\mathrm{Kos}"] & \cdots \\
& & \bigwedge^a K \otimes \bigwedge^g G \otimes \bigwedge^g G^* \arrow[r, "\cong"'] & \bigwedge^a K \arrow[ur, "\bigwedge^a i"] & &
\end{tikzcd}$$

and everything is fine there.

We prove (a) by induction on $f-g$.

Marching orders for Tuesday April 4.

Theorem. *Let R be a Noetherian ring, $\phi : F \to G$ be a homomorphism of free R-modules with*

$$\operatorname{rank} G = g \le f = \operatorname{rank} F.$$

If $f - g + 1 \le \operatorname{grade} I_g(\phi)$, then $\mathcal{C}^q$ is acyclic for $-1 \le q \le f - g + 1$.

Proof. **Step 1.** Apply the acyclicity lemma. It suffices to prove the result when R is local and ϕ surjective.

Step 2. Assume R is local and ϕ is surjective. Let $K = \ker \phi$. It follows that K is a direct summand of F; that is, $F = K \oplus L$ for some submodule L of F. Let $i : K \to F$ and $\pi : F \to K$ be the homomorphisms which correspond to $F = K \oplus L$. To complete the proof it suffices to prove that

$$0 \to \textstyle\bigwedge^j K \otimes \operatorname{Sym}_0 G \xrightarrow{\bigwedge^j i} \bigwedge^j F \otimes \operatorname{Sym}_0 G \xrightarrow{\operatorname{Kos}_\phi} \cdots \xrightarrow{\operatorname{Kos}_\phi} \bigwedge^0 F \otimes \operatorname{Sym}_j G \to 0$$

is split exact for all non-negative integers j.

The proof is by induction on $f - g$.

• If $f = g$, then $\bigwedge^\bullet(F \otimes_R \operatorname{Sym}_\bullet G)$ is a $\operatorname{Sym}_\bullet G$-resolution of R because

$$F \otimes_R \operatorname{Sym}_\bullet G \xrightarrow{\phi\otimes 1} G \otimes_R \operatorname{Sym}_\bullet G \xrightarrow{\text{mult}} \operatorname{Sym}_\bullet G$$

ultimately is

$$[T_1, \dots, T_g],$$

(if one thinks of $\operatorname{Sym}_\bullet G$ as $R[T_1, \dots, T_g]$.) So, every strand of $\bigwedge^\bullet(F \otimes_R \operatorname{Sym}_\bullet G)$, except

$$0 \to \textstyle\bigwedge^0 F \otimes_R \operatorname{Sym}_0 G \to 0,$$

is split exact. Of course,

$$0 \to \underbrace{\textstyle\bigwedge^0 K \otimes \operatorname{Sym}_0 G}_{R} \to \bigwedge^0 F \otimes \operatorname{Sym}_0 G \to 0$$

is also split exact.

• If $g < f$, then select a basis element e of K. Write $K = K' \oplus Re$ and $F = F' \oplus Re$, for $F' = K' \oplus \ker \pi$. Observe that

$$\textstyle\bigwedge^j F = \bigwedge^j F' \oplus (\bigwedge^{j-1} F' \otimes Re)$$

and

$$0 \to \textstyle\bigwedge^j F' \to \bigwedge^j F \to \bigwedge^{j-1} F' \otimes Re \to 0$$

is a short exact sequence of modules. Consider the short exact sequence of complexes:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bigwedge^j K' \otimes \mathrm{Sym}_0 G \arrow[r, "\bigwedge^j i"] \arrow[d] & \bigwedge^j F' \otimes \mathrm{Sym}_0 G \arrow[r, "\mathrm{Kos}_\phi"] \arrow[d] & \cdots \arrow[r, "\mathrm{Kos}_\phi"] & \bigwedge^1 F' \otimes \mathrm{Sym}_{j-1} G \arrow[r, "\mathrm{Kos}_\phi"] \arrow[d] & \bigwedge^0 F' \otimes \mathrm{Sym}_j G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigwedge^j K \otimes \mathrm{Sym}_0 G \arrow[r, "\bigwedge^j i"] \arrow[d] & \bigwedge^j F \otimes \mathrm{Sym}_0 G \arrow[r, "\mathrm{Kos}_\phi"] \arrow[d] & \cdots \arrow[r, "\mathrm{Kos}_\phi"] & \bigwedge^1 F \otimes \mathrm{Sym}_{j-1} G \arrow[r, "\mathrm{Kos}_\phi"] \arrow[d] & \bigwedge^0 F \otimes \mathrm{Sym}_j G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigwedge^{j-1} K' \otimes Re \otimes \mathrm{Sym}_0 G \arrow[r, "\bigwedge^j i"] \arrow[d] & \bigwedge^{j-1} F' \otimes Re \otimes \mathrm{Sym}_0 G \arrow[r, "\mathrm{Kos}_\phi"] \arrow[d] & \cdots \arrow[r, "\mathrm{Kos}_\phi"] & \bigwedge^0 F \otimes Re \otimes \mathrm{Sym}_{j-1} G \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & & 0 & &
\end{tikzcd}$$

The top and bottom complexes are split exact by induction. Use the long exact sequence of homology to conclude that the middle complex is also split exact. □

4.K. **Depth Sensitivity and perfection.**

Goal 4.13. *Suppose that R is a commutative Noetherian ring, M is a finitely generated R-module of finite projective dimension and*

$$\operatorname{grade} \operatorname{ann}_R M = \operatorname{pd}_R M.$$

Let F be a free resolution of M of length equal to $\operatorname{pd}_R M$ and let S be a commutative Noetherian R-algebra. We will show that

if $(\operatorname{ann}_R M)S$ *is a proper ideal of S and*

(4.13.1) $$\operatorname{grade} \operatorname{ann}_R M \le \operatorname{grade}(\operatorname{ann}_R M)S,$$

then $F \otimes_R S$ is a free resolution of $M \otimes_R S$.

We will also show that it is always the case that

(4.13.2) *the final* $\operatorname{grade}(\operatorname{ann}_R M)S$ *maps from $F \otimes_R S$ form an acyclic complex.*

Remarks. (a) It is always true that $\operatorname{grade} \operatorname{ann}_R M \le \operatorname{pd}_R M$. It is possible that I only proved that $\operatorname{grade} I \le \operatorname{pd}_R R/I$. The main step in the proof is

$$\operatorname{grade} I = \min\{i \mid \operatorname{Ext}^i(R/I, R) \ne 0\}.$$

The same argument shows

$$\operatorname{grade} \operatorname{ann} M = \min\{i \mid \operatorname{Ext}^i(M, R) \ne 0\};$$

and this yields $\operatorname{grade} \operatorname{ann}_R M \le \operatorname{pd}_R M$.

(b) If $\operatorname{grade} \operatorname{ann}_R M = \operatorname{pd}_R M$, then M is a perfect R-module.

(c) If $x_1, \dots, x_g$ is a regular sequence on R, then $R/(x_1, \dots, x_g)$ is a perfect R-module. In the context of the generalized Eagon-Northcott complexes, then $\mathrm{H}^0(\mathcal{C}^q)$ is a perfect R-module for $-1 \le q \le f-g+1$ provided $f-g+1 \le \operatorname{grade} I_{f-g+1}(\phi)$. (This includes the Hilbert-Burch situation.) If I is generated by the maximal order Pfaffians of an odd-sized alternating matrix and $3 \le \operatorname{grade} I$, then R/I is a perfect R-module.

(d) The property (4.13.1) is called the Transfer of perfection [1, chapter 3A] or the persistence of perfection [4, section 6].

(e) If a complex F satisfies property 4.13.2, then one says that F exhibits depth sensitivity.

4.14. Proof of Goal 4.13. In light of WMACE, the only thing to show is that if M is a finitely generated perfect R-module and F is a free resolution of M of length $\operatorname{pd}_R M$, then $\sqrt{\operatorname{ann} M} = \sqrt{I(f_j)}$.

Recall that

$$\sqrt{\operatorname{ann} M} = \bigcap_{\operatorname{ann} M \subseteq P} P.$$

It suffices to prove that if $P \in \operatorname{Spec} R$, then

$$\operatorname{ann} M \subseteq P \iff I(f_j) \subseteq P$$

for all j.

• If $\operatorname{ann} M \not\subseteq P$, then $M_P = 0$; F_P is split exact, $I(f_j)_P = R_P$ and $I(f_j) \not\subseteq P$. (We used the fact that $I(f_j)$ contains a regular element (or a unit) of R; hence, the rank of f_j is equal to the rank of $(f_j)_P$ for all prime ideals P.)

• If $\operatorname{ann} M \subseteq P$, then

$$\operatorname{grade} \operatorname{ann} M \le \operatorname{grade}(\operatorname{ann} M)_P =_* \operatorname{grade}(\operatorname{ann}(M_P)) \le \operatorname{pd}_{R_P} M_P \le \operatorname{pd}_R M = \operatorname{grade} \operatorname{ann} M.$$

The last equality is the hypothesis. The equality $=_*$ holds because $(\operatorname{ann} M)_P = \operatorname{ann}(M_P)$. The inclusion $\subseteq$ holds automatically. The inclusion $\supseteq$ uses the fact that M is finitely generated: if $r \in R$ kills each generator of M in M_P, then there exists $s \in R \setminus P$ such that sr kills each generator of M.

Thus, $\operatorname{pd}_{R_P} M_P = \operatorname{pd}_R M$ and $I(f_j) \subseteq P$ for all j.

4.15. Application of 4.13. Let R be a commutative Noetherian ring, $I = (a_1, \dots, a_f)$ be an ideal of I, and K be the Koszul complex on $a_1, \dots, a_f$. If g is an integer with $g \le \operatorname{grade} I$, then the final g maps of K form an acyclic complex. That is,

$$H_i(K) = 0 \quad \text{for } f - g + 1 \le i \le f.$$

I need to see the following picture before I can sign-off on the above claim

$$K_f \xrightarrow[1]{} K_{f-1} \xrightarrow[2]{} K_{f-2} \xrightarrow[3]{} \cdots \xrightarrow[g-1]{} K_{f-g+1} \xrightarrow[g]{} K_{f-g}.$$

5. THE GRADE OF $I_g(\phi)$ WHEN ϕ IS A MATRIX OF VARIABLES; THE RESOLUTION OF $(r_1, \dots, r_\ell)^g$ WHEN $r_1, \dots, r_\ell$ IS A REGULAR SEQUENCE; AND HOW TO READ SOCLE DEGREES FROM A HOMOGENEOUS RESOLUTION.

5.A. **The grade of $I_g(\phi)$ when ϕ is a matrix of variables.**

Example 5.1. Consider the $g \times f$ matrix

$$\phi = \begin{bmatrix} a_1 & a_2 & \cdots & a_\ell & 0 & 0 & 0 \\ 0 & a_1 & a_2 & \cdots & a_\ell & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & 0 & 0 & a_1 & a_2 & \cdots & a_\ell \end{bmatrix}. \tag{5.1.1}$$

- Observe that $f = \ell + g - 1$.

- Observe that $(a_1, \dots, a_\ell) \subseteq \sqrt{I_g(\phi)}$. Indeed, a_1^g is the determinant of the first g columns; hence $a_1 \in \sqrt{I_g(\phi)}$. The determinant of columns $2, \dots, g+1$ is congruent to a_2^g, mod (a_1); hence $a_2 \in \sqrt{I_g(\phi)}$. Etc.

- Observe that if $R = R_0[\{x_{i,j} \mid 1 \le i \le g \text{ and } 1 \le j \le f]$, and ϕ is the $g \times f$ matrix $(x_{i,j})$, then $f - g + 1 \le \operatorname{grade} I_g(\phi)$. We use the result from the footnote at the end of Chapter 1: If $R = \bigoplus_{0 \le i} R_i$ is a commutative Noetherian ring, $f_1, \dots f_n, x$ are homogeneous elements of positive degree in R, with $\bar{f}_1, \dots \bar{f}_n$ a regular sequence in $\bar{R} = R/(x)$, then $f_1, \dots f_n$ is a regular sequence in R.

- The second goal in the section is to prove that $I_g(\phi) = (a_1, \dots, a_\ell)^g$. I suppose one could do this directly, but I would much rather use trickery. (The trickery works provided the ring contains a filed.) Of course, the ultimate point is that the Eagon-Northcott complex for ϕ (from 5.1.1) resolves $R/(a_1, \dots, a_\ell)^g$, when $a_1, \dots, a_\ell$ is a regular sequence in R.

5.B. **The convention for shifting the degree of a graded module.** If $M = \bigoplus M_i$ is a graded module and a is an integer, then $M(a)$ is a new graded module with

$$M(a)_i = M_{a+i}.$$

For example the element 1 has degree one in $R(-1)$ and therefore

$$R(-1) \xrightarrow{x} R$$

is a homogeneous map of degree zero for all $x \in R_1$.

If the entries of ϕ are linear of degree 1, then the Eagon-Northcott complex looks like:

$$0 \to \underbrace{\textstyle\bigwedge^f F \otimes \bigwedge^g G^* \otimes D_{f-g} G^*}_{R(-f)^{\binom{(f-g)+(g-1)}{g-1}}} \to \cdots \to \underbrace{\textstyle\bigwedge^g F \otimes \bigwedge^g G^* \otimes D_0 G^*}_{R(-g)^{\binom{f}{g}}} \to \underbrace{\textstyle\bigwedge^0 F \otimes \operatorname{Sym}_0 G}_{R}$$

Theorem. *Let $R = \boldsymbol{k}[x_1, \dots, x_n]$ be a standard-graded polynomial ring over a field $\boldsymbol{k}$ (that is each variable has degree one) and M be a homogeneous Artinian R-module. (In particular,*

M is finitely generated, has depth zero and projective dimension n.) If the minimal graded resolution of M is

$$0 \to \bigoplus_i R(-b_i) \to \dots,$$

then the socle of M has a homogeneous basis with elements of degree $\{b_i - n\}$.

The lecture of April 10, 2018.

Today's first goal is to make sense of

Theorem. *Let $R = \boldsymbol{k}[x_1, \dots, x_n]$ be a standard-graded polynomial ring over a field $\boldsymbol{k}$ (that is each variable has degree one) and M be a homogeneous Artinian R-module. (In particular, M is finitely generated, has depth zero and projective dimension n.) If the minimal graded resolution of M is*

$$0 \to \bigoplus_i R(-b_i) \to \dots,$$

then the socle of M has a homogeneous basis with elements of degree $\{b_i - n\}$.

The second goal is to apply the Theorem to show that $I_g(\phi) = (x_1, \dots, x_\ell)^g$ where ϕ is the $g \times f$ matrix

$$\phi = \begin{bmatrix} x_1 & x_2 & \cdots & x_\ell & 0 & 0 & 0 \\ 0 & x_1 & x_2 & \cdots & x_\ell & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & 0 & 0 & x_1 & x_2 & \cdots & x_\ell \end{bmatrix},$$

$f = \ell + g - 1$, and R is the polynomial ring $\boldsymbol{k}[x_1, \dots, x_\ell]$.

We start with two examples of the Theorem. Consider the resolution

$$0 \to R(-3)^2 \xrightarrow{\begin{bmatrix} x & 0 \\ y & x \\ 0 & y \end{bmatrix}} R(-2)^3 \xrightarrow{\begin{bmatrix} y^2 & -xy & x^2 \end{bmatrix}} R.$$

Observe that the socle of $A = R/(x, y)^2$ has socle $\boldsymbol{k}x \oplus \boldsymbol{k}y \cong \boldsymbol{k}(-1)^2$. The Theorem promises that the socle of A is $\boldsymbol{k}(-(3-2))^2$.

Consider the resolution

$$0 \to R(-5) \xrightarrow{\begin{bmatrix} y^2 \\ -xz \\ xy + z^2 \\ -yz \\ x^2 \end{bmatrix}} R(-3)^5 \xrightarrow{\begin{bmatrix} 0 & y & 0 & 0 & z \\ -y & 0 & x & z & 0 \\ 0 & -x & 0 & y & 0 \\ 0 & -z & -y & 0 & x \\ -z & 0 & 0 & -x & 0 \end{bmatrix}} R(-2)^5 \xrightarrow{\begin{bmatrix} y^2 & -xz & xy + z^2 & -yz & x^2 \end{bmatrix}} R.$$

Observe that $A = R/(y^2, -xz, xy + z^2, -yz, x^2)$ has socle $\boldsymbol{k}z^2 \cong \boldsymbol{k}(-2)^1$. The Theorem promises that the socle of A is isomorphic to $\boldsymbol{k}(-(5-3))^1$.

Proof of Theorem. On the one hand $\operatorname{Tor}_n^R(M, \boldsymbol{k}) = \bigoplus_i \boldsymbol{k}(-b_i)$, which has a homogeneous basis whose degrees are $\{b_i\}$. On the other hand,

$$\operatorname{Tor}_n^R(M, \boldsymbol{k}) = \mathrm{H}_n(M \otimes_R \text{Koszul complex}) = \ker \left(M(-n) \xrightarrow{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} M(-(n-1)) \right) = \operatorname{socle} M(-n).$$

If the socle degrees are $\{d_j\}$ then the generator degress of $\operatorname{socle} M(-n)$ are $\{d_j + n\}$. It follows that the socle degrees are $\{b_i - n\}$. $\square$

Now we apply the Theorem to $A = \boldsymbol{k}[x_1, \dots, x_\ell]/I_g(\phi)$. We know that $\mathcal{C}^0_\phi$ resolves A. The entries of ϕ are linear of degree 1, therefore, the Eagon-Northcott complex looks like:

$$0 \to \underbrace{\bigwedge^f F \otimes \bigwedge^g G^* \otimes D_{f-g} G^*}_{R(-f)^{\binom{(f-g)+(g-1)}{g-1}}} \to \cdots \to \underbrace{\bigwedge^g F \otimes \bigwedge^g G^* \otimes D_0 G^*}_{R(-g)^{\binom{f}{g}}} \to \underbrace{\bigwedge^0 F \otimes \operatorname{Sym}_0 G}_{R}$$

Apply the Theorem to see that the socle of $A = \boldsymbol{k}[x_1, \dots, x_\ell]/I_g(\phi)$ lives in degree $f - \ell = g - 1$ and has vector space dimension $\binom{(f-g)+(g-1)}{g-1}$. Recall that the dimension of $\boldsymbol{k}[x_1, \dots, x_\ell]_{g-1}$ is $\binom{\ell-1+g-1}{g-1} = \binom{(f-g)+(g-1)}{g-1}$. Thus the socle of A is precisely $\mathfrak{m}^{g-1}$ and $I_g = \mathfrak{m}^g$ for $\mathfrak{m} = (x_1, \dots, x_\ell)$.

6. THE LASCOUX RESOLUTION.

I am taking this discussion from [10, Section 6.1].

Let $\boldsymbol{k}$ be a field of **characteristic zero**, $(x_{i,j})$ be a $g \times f$ matrix of indeterminates, r be an integer with $0 \le r < g \le f$, and R be the polynomial ring $\boldsymbol{k}[\{x_{i,j}\}]$. We describe the resolution of $R/I_{r+1}((x_{i,j}))$ by free R-modules.

Let F and G be free $\boldsymbol{k}$-modules of rank f and g respectively. Recall that $\operatorname{Hom}_{\boldsymbol{k}}(F,G) = F^* \otimes_{\boldsymbol{k}} G$. We think of R as $\operatorname{Sym}_\bullet^{\boldsymbol{k}}(F \otimes G^*)$. Keep in mind that R is the coordinate ring for the affine space $X = F^* \otimes_k G$ in the sense that each element of $F \otimes G^*$ represents a (coordinate) map from X to $\boldsymbol{k}$. Of course, "$I_{r+1}((x_{i,j}))$" is the ideal which vanishes on the subvariety

$$Y_r = \{\phi \in \operatorname{Hom}(F,G) \mid \operatorname{rank} \phi \le r\}$$

of X.

Pick it up here on April 12:

- $\boldsymbol{k}$ is a field of characteristic zero.
- F and G are vector spaces over $\boldsymbol{k}$ of dimension f and g respectively.
- $X = F^* \otimes_k G \cong \boldsymbol{k}^{fg}$
- Let r be an integer with $0 \le r \le g-1$.
- $Y_r = \{\phi \in \underbrace{\operatorname{Hom}(F,G)}_{\cong X} \mid \operatorname{rank} \phi \le r\}$
- $R = \operatorname{Sym}^{\boldsymbol{k}}(F \otimes G^*)$
- We resolve $R/I(Y_r)$ by free R-modules.
- Of course, $R/I(Y_r)$ is a coordinate free way of saying
$$\frac{\boldsymbol{k}[(x_{i,j})_{g\times f}]}{I_{r+1}(x_{ij})}.$$

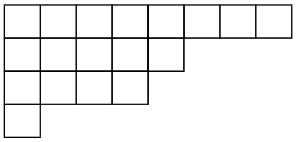
The group $\mathcal{G} = \operatorname{GL}(F) \times \operatorname{GL}(G^*)$ acts on the ring R, on the ring $R/I(Y_r)$, and on the resolution of $R/I(Y_r)$ by free R-modules. The characteristic of $\boldsymbol{k}$ is zero; so every finite dimensional $\mathcal{G}$-module is the direct sum of modules of the form:

$$\text{simple } \operatorname{GL}(F)\text{-module} \otimes \text{simple } \operatorname{GL}(G^*)\text{-module}.$$

The simple $\operatorname{GL}(F)$-modules are in a one-to-one correspondence with the set of partitions

$$\lambda : \quad \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n,$$

with $\lambda_1 \le f$ and the λ_i are positive integers. The above partition is a partition of $|\lambda| = \sum_i \lambda_i$. One might draw the above partition as the following Young tableaux:



with λ_i boxes in the row i. The module $L_{(\lambda_1,\lambda_2,\cdots,\lambda_n)}F$ is equal

$$\frac{\bigwedge^{\lambda_1} F\otimes\cdots\otimes\bigwedge^{\lambda_n} F}{\text{complicated submodule}}.$$

In particular, $L_{(t)}F = \bigwedge^t F$ and $L_{\underbrace{(1,\dots,1)}_{t}}F = \operatorname{Sym}_t F$. One usually writes 1^t in place of $\underbrace{(1,\dots,1)}_{t}$. The vector space $L_{(\lambda_1,\lambda_2,\cdots,\lambda_n)}F$ has dimension equal to the number of ways to fill in the Young tableau using the numbers from 1 to f with each row strictly increasing and each column non-decreasing. In fact, if $e_1,\cdots,e_f$ is a basis for F, then one builds a basis for $L_\lambda F$ using the above recipe.

Example 6.1. If $e_1,\dots,e_4$ is a basis for F, then the basis for $L_{(3,1)}F$ is represented by

$$\begin{array}{cccc}
e_1\wedge e_2\wedge e_3\otimes e_1 & e_1\wedge e_2\wedge e_3\otimes e_2 & e_1\wedge e_2\wedge e_3\otimes e_3 & e_1\wedge e_2\wedge e_3\otimes e_4\\
e_1\wedge e_2\wedge e_4\otimes e_1 & e_1\wedge e_2\wedge e_4\otimes e_2 & e_1\wedge e_2\wedge e_4\otimes e_3 & e_1\wedge e_2\wedge e_4\otimes e_4\\
e_1\wedge e_3\wedge e_4\otimes e_1 & e_1\wedge e_3\wedge e_4\otimes e_2 & e_1\wedge e_3\wedge e_4\otimes e_3 & e_1\wedge e_3\wedge e_4\otimes e_4\\
 & e_2\wedge e_3\wedge e_4\otimes e_2 & e_2\wedge e_3\wedge e_4\otimes e_3 & e_2\wedge e_3\wedge e_4\otimes e_4
\end{array}$$

Theorem. [Lascoux] *If the characteristic of $\boldsymbol{k}$ is zero then the minimal homogeneous resolution of $R/I(Y_r)$ by free R-modules has the form*

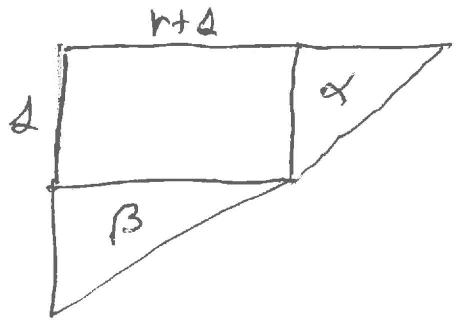
$$0\to F_{(f-r)(g-r)}\to\cdots\to F_i\to\cdots\to F_0\to 0,$$

where

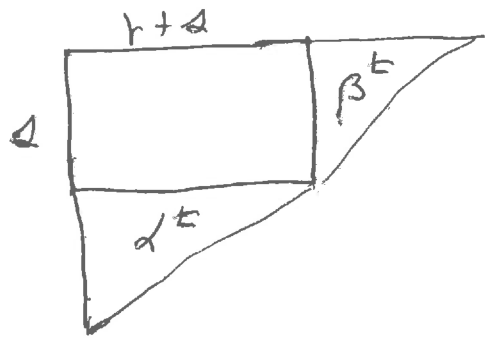
$$F_i = \bigoplus_{\left\{(s,\alpha,\beta)\;\middle|\;\begin{array}{l} s \text{ is a non-negative integer}\\ \alpha \text{ and } \beta \text{ are partitions}\\ i=s^2+|\alpha|+|\beta|\\ \alpha\subseteq(f-r-s)^s\\ \beta\subseteq(s)^{g-r-s}\end{array}\right\}} L_{P_1(s,\alpha,\beta)}F\otimes_{\boldsymbol{k}} L_{P_2(s,\alpha,\beta)}G^*\otimes_{\boldsymbol{k}}\operatorname{Sym}^{\boldsymbol{k}}_{\bullet}(F\otimes_{\boldsymbol{k}} G^*),$$

where the pictures for $P_1(s,\alpha,\beta)$ and $P_2(s,\alpha,\beta)$ are given on the next page. In words, the Young diagram for $P_1(s,\alpha,\beta)$ is obtained by putting the Young diagram for α to the right of an $r\times s$ rectangle and putting the Young diagram for β below the $r\times s$ rectangle and the Young diagram for $P_2(s,\alpha,\beta)$ is obtained by putting $\beta^{\text{transpose}}$ to the right of an $r\times s$ rectangle and putting $\alpha^{\text{transpose}}$ below the $r\times s$ rectangle.

The notation p^q represents the partition $\underbrace{(p,\dots,p)}_{q}$. One writes $\lambda\subseteq\mu$ for partitions $\lambda=(\lambda_1,\dots,\lambda_\ell)$ and $\mu=(\mu_1,\dots,\mu_m)$ to mean that $\lambda_i\le\mu_i$ for all i. If λ is the partition $(\lambda_1,\dots,\lambda_\ell)$, then $|\lambda|=\sum_i\lambda_i$. If the Young diagram for α has α_i boxes in **ROW** i, then the Young diagram for $\alpha^{\text{transpose}}$ has α_i boxes in **COLUMN** i. Of course, $R=\operatorname{Sym}^{\boldsymbol{k}}_{\bullet}(F\otimes_{\boldsymbol{k}} G^*)$.



$P_1(\Delta,\alpha,\beta)$



$P_2(\Delta,\alpha,\beta)$

Example. Let us see what the Lascoux complex is when $r = g - 1$. (Of course, we expect to see the Eagon-Northcott complex.)

The parameter $s = 0$ only contibutes when $i = 0$ and the contribution is

$$L_{(0)}F \otimes_{\boldsymbol{k}} L_{(0)}G^* \otimes_{\boldsymbol{k}} R = R.$$

There is no contribution for $2 \leq s$ because if $\beta \subseteq (s)^{g-(g-1)-2}$, then β does not exist.

Consider $s = 1$. Take $\alpha \subseteq (f-(g-1)-1)^1 = (f-g)^1$ and $\beta \subseteq (1)^{g-(g-1)-1} = (1)^0$. Recall that

$$i = s^2 + |\alpha| + |\beta| = 1 + |\alpha| + 0$$

so $\alpha = (i-1)$, $P_1(1,(i-1),\alpha,\beta)$ is an $s \times (r+s) = 1 \times g$ rectangle with an $(i-1) \times 1$ rectangle on its right:

$$P_1(1,(i-1),\alpha,\beta) = (g+i-1).$$

Also $P_2(1,\alpha,\beta)$ is a $1 \times g$ rectangle above an $1 \times (i-1)$ rectangle. Hence

$$P_2(1,\alpha,\beta) = (g, 1^{i-1}).$$

It follows that $F_0 = R$ and

$$\begin{aligned} F_i &= L_{(g+i-1)}F \otimes L_{(g,1^{i-1})}G^* \otimes R = \bigwedge^{g+i-1} F \otimes \bigwedge^g G^* \otimes \operatorname{Sym}_{i-1} G^* \otimes R \\ &= \bigwedge^{g+i-1} F \otimes \bigwedge^g G^* \otimes D_{i-1}G^* \otimes R, \end{aligned}$$

for $1 \leq i \leq \underbrace{(f-(g-1))(g-(g-1))}_{f-g+1}$. The last equality holds because the characteristic of $\boldsymbol{k}$ is zero. Of course, this **IS** our old friend

$$\cdots \to \bigwedge^{g+1} F \otimes \bigwedge^g G^* \otimes D_1G^* \to \bigwedge^g F \otimes \bigwedge^g G^* \otimes D_0G^* \to \bigwedge^0 F \otimes \operatorname{Sym}_0 G.$$

Example. Lets see what the Lascoux complex is when $r = 0$. In this case one is resolving $k[\{x_{i,j}\}]/(x_{i,j})$. The resolution should be the Koszul complex on $\{x_{i,j}\}$.

The contribution when $s = 0$ involves $\alpha \subseteq (f-r-s)^s = (f)^0$ and $\beta \subseteq (s)^{g-r-s} = (0)^g$. Thus, $\alpha = (0)$, $\beta = (0)$ and the contribution is $F_0 = L_{(0)}F \otimes_{\boldsymbol{k}} L_{(0)}G^* \otimes_{\boldsymbol{k}} R = R$. For $1 \leq i$,

$$F_i = \bigoplus_{\left\{ (s,\alpha,\beta) \,\middle|\, \begin{array}{l} s \text{ is a non-negative integer} \\ \alpha \text{ and } \beta \text{ are partitions} \\ i = s^2 + |\alpha| + |\beta| \\ \alpha \subseteq (f-r-s)^s \\ \beta \subseteq (s)^{g-r-s} \end{array} \right\}} L_{P_1(s,\alpha,\beta)}F \otimes_{\boldsymbol{k}} L_{P_2(s,\alpha,\beta)}G^* \otimes_{\boldsymbol{k}} \operatorname{Sym}^{\boldsymbol{k}}_{\bullet}(F \otimes_{\boldsymbol{k}} G^*).$$

As (s,α,β) roam over all legal values the pair

$$(P_1(s,\alpha,\beta), P_2(s,\alpha,\beta))$$

roams over $(\lambda, \lambda^{\text{transpose}})$ with λ a partition of i with at most f columns and $\lambda^{\text{transpose}}$ has at most g columns. **DRAW THE PICTURE.** It is obvious. So,

$$F_i = \sum_{|\lambda|=i} L_\lambda F \otimes L_{\lambda^{\text{transpose}}}G^*$$

and this is the Cauchy formula for $\bigwedge^i(F \otimes G^*)$. I posted a link to a Math Stack Exchange question about an exterior power of a tensor product; it gives a reference to Fulton-Harris, Representation Theory, exercise 6.11.

Example. Let us record the Lascoux complex for the ring $R/I_{g-1}(M)$, where M is a $g \times g$ matrix of variables. The ring $R/I_{g-1}(M)$ is a codimension 4 Gorenstein quotient of R; its graded Betti numbers are

$$0 \to R(-2g) \to R(-(g+1))^{g^2} \to R(-g)^{2g^2-2} \to R(-(g-1))^{g^2} \to R \to 0$$

This resolution was first worked out by Gulliksen-Negård. We take $f = g$ and $r = g-2$:

$$\begin{aligned} &i = s^2 + |\alpha| + |\beta| \\ &\alpha \subseteq (g-(g-2)-s)^s = (2-s)^s \\ &\beta \subseteq (s)^{g-(g-2)-s} = (s)^{2-s} \end{aligned}$$

- When $s = 0$, then $\alpha \subseteq (2)^0$ and $\beta \subseteq (0)^2$; so $\alpha = \beta = (0)$.
- When $s = 1$, then $\alpha \subseteq (1)^1$ and $\beta \subseteq (1)^1$; so

$$\alpha = (0) \quad \text{or} \quad \alpha = (1) \qquad \text{and} \qquad \beta = (0) \quad \text{or} \quad \beta = (1).$$

- When $s = 2$, then $\alpha \subseteq (0)^2$ and $\beta \subseteq (2)^0$; so $\alpha = \beta = (0)$.

The resolution has six summands. We tack α, β and their transposes onto an $s \times (s+g-2)$ rectangle

s	α	β	i	$L_{P_1(s,\alpha,\beta)}F \otimes L_{P_2(s,\alpha,\beta)}G^*$	other name
0	(0)	(0)	0	$L_{(0)}F \otimes L_{(0)}G^*$	$= \boldsymbol{k}$
1	(0)	(0)	1	$L_{(g-1)}F \otimes L_{(g-1)}G^*$	$= \bigwedge^{g-1} F \otimes \bigwedge^{g-1} G^*$
1	(0)	(1)	2	$L_{(g-1,1)}F \otimes L_{(g)}G^*$	$= \operatorname{coker}(\bigwedge^g F \xrightarrow{\Delta} \bigwedge^{g-1} F \otimes \bigwedge^1 F) \otimes \bigwedge^g G^*$
1	(1)	(0)	2	$L_{(g)}F \otimes L_{(g-1,1)}G^*$	$= \bigwedge^g F \otimes \operatorname{coker}(\bigwedge^g G^* \xrightarrow{\Delta} \bigwedge^{g-1} G^* \otimes \bigwedge^1 G^*)$
1	(1)	(1)	3	$L_{(g,1)}F \otimes L_{(g,1)}G^*$	$= \bigwedge^g F \otimes F \otimes \bigwedge^g G^* \otimes G^*$
2	(0)	(0)	4	$L_{(g)}F \otimes L_{(g)}G^*$	$= \bigwedge^g F \otimes \bigwedge^g G^*$

The resolution is

$L_{(g)}F \otimes L_{(g)}G^* \otimes R$				
	$L_{(g,1)}F \otimes L_{(g,1)}G^* \otimes R$	$L_{(g-1,1)}F \otimes L_{(g)}G^* \otimes R$ $\oplus$ $L_{(g)}F \otimes L_{(g-1,1)}G^* \otimes R$	$L_{(g-1)}F \otimes L_{(g-1)}G^* \otimes R$	
				$L_{(0)}F \otimes L_{(0)}G^* \otimes R$

The vertical lines in the above picture represent where the differentials live. When I drew the modules, I seperated the linear strands. The linear strands are tagged with the different values of s. The maps in each linear strand are linear. The maps from one linear strand to another have higher degree.

Example. We describe the end of the resolution. This will give $\operatorname{pd}_R(R/I(Y_r))$, the minimal number of generators of the canonical module of $R/I(Y_r)$ (once one realizes that $R/I_r(Y)$ is Cohen-Macaulay) (and also information about the degrees of the generators of this module if we keep track of degrees), the dimension of the socle of a zero-dimensional specialization $R/I(Y_r)$ (again one should prove that $R/I(Y_r)$ is Cohen-Macaulay or is a perfect R-module before specializing) and also the degrees of these socle generators if we keep track of degrees.

At any rate, I do a Math 141 problem to maximize

$$i = s^2 + |\alpha| + |\beta|,$$

for $\alpha \subseteq (f-r-s)^s$ and $\beta \subseteq (s)^{(g-r-s)}$. It is clear that i is maximized when $\alpha = (f-r-s)^s$ and $\beta = (s)^{(g-r-s)}$. So we maximize

$$i(s) = s^2 + (f-r-s)s + (s)(g-r-s)$$

subject to the constraints

$$0 \le f-r-s, \quad 0 \le s, \quad 0 \le g-r-s.$$

Maximize

$$i(s) = s^2 + (f-r-s)s + (s)(g-r-s) \quad \text{with } 0 \le s \le g-r.$$

Observe that

$$i(s) = fs - 2rs + gs - s^2;$$

hence

$$i'(s) = f + g - 2r - 2s$$

and $i'(s) = 0$ when $s = (f+g)/2 - r$. Of course $g - r \le (f+g)/2 - r$ so either $i'(s) = 0$ at an end point or at a point not in the domain. Thus the max and min of i occur at the endpoints and $i(0) = 0$ and

$$i(g-r) = (g-r)^2 + (f-r-(g-r))(g-r) = (f-g)(g-r).$$

The projective dimension of $R/I(Y_r)$ is $(f-g)(g-r)$ and the final module in the Lascoux resolution has $s = g-r$, $\alpha = (f-g)^{(g-r)}$ and $\beta = (0)$. **Draw the picture.** We see that $P_1(s,\alpha,\beta) = f^{g-r}$ and $P_2(s,\alpha,\beta) = (g^{(g-r)}, (g-r)^{(f-g)})$.

It is reasonable to ask, "When is $\dim_{\boldsymbol{k}} F_{(f-r)(g-r)} = 1$?" (This is equivalent to asking when is $R/I_r(Y)$ Gorenstein (since $R/I_r(Y)$ is Cohen-Macaulay)). Well,

$$\dim_{\boldsymbol{k}} F_{(f-r)(g-r)} = \dim_{\boldsymbol{k}} L_{f^{g-r}}F \cdot \dim L_{(g^{(g-r)},(g-r)^{(f-g)})}G^* = \dim L_{(g-r)^{(f-g)}}G^*$$

and this is 1 if and only if $f = g$ or $r = 0$.

6.A. Define $L_\lambda F$.

Definition. If F is a finite dimensional vector space and $\lambda = (\lambda_1, \dots, \lambda_n)$ is a partition, then the Schur module $L_\lambda F$ is equal to

$$L_\lambda F = \frac{\bigwedge^{\lambda_1} F \otimes \cdots \otimes \bigwedge^{\lambda_n} F}{\sum\limits_{a=1}^{n-1} \bigwedge^{\lambda_1} F \otimes \cdots \otimes \bigwedge^{\lambda_{a-1}} F \otimes R_{a,a+1}F \otimes \bigwedge^{\lambda_{a+2}} F \otimes \cdots \otimes \bigwedge^{\lambda_n} F},$$

where $R_{a,a+1}F$ is equal to

$$\sum_{u+v<\lambda_{a+1}} \operatorname{im}\left(\bigwedge^u F \otimes \bigwedge^{\lambda_a - u + \lambda_{a+1} - v} F \otimes \bigwedge^v F \xrightarrow{1\otimes\Delta\otimes 1} \bigwedge^u F \otimes \bigwedge^{\lambda_a - u} F \otimes \bigwedge^{\lambda_{a+1}-v} F \otimes \bigwedge^v F \xrightarrow{\text{mult}\otimes\text{mult}} \bigwedge^{\lambda_a} F \otimes \bigwedge^{\lambda_{a+1}} F\right).$$

Examples. (a) $L_t F = \bigwedge^t F$.

(b) $L_{1^t}F = \operatorname{Sym}_t F$, because

$$L_{1^t}F = \frac{F^{\otimes t}}{\left(\begin{array}{c} (\{(f_1 \otimes f_2 - f_2 \otimes f_1) \otimes F^{\otimes t-2} \mid f_1, f_2 \in F\}) \\ +(\{F \otimes (f_1 \otimes f_2 - f_2 \otimes f_1) \otimes F^{\otimes t-3} \mid f_1, f_2 \in F\}) \\ \vdots \\ +(\{F^{\otimes t-2} \otimes (f_1 \otimes f_2 - f_2 \otimes f_1) \mid f_1, f_2 \in F\}) \end{array}\right)} = \operatorname{Sym}_t F.$$

(c) If $p+q \neq 0$, then

$$\cdots \to \bigwedge^p F \otimes \operatorname{Sym}_q F \xrightarrow{\operatorname{Kos}_{\mathrm{id}}} \bigwedge^{p-1} F \otimes \operatorname{Sym}_{q+1} F \to \cdots \qquad (6.1.1)$$

is split exact. (We proved this when we proved that the Eagon-Northcott complex is exact. This is the base case of the proof that if ϕ is surjective, then the right side of the Eagon-Northcott complex is exact.) Of course, if $p+q=0$, then the complex is

$$\cdots \to 0 \to 0 \to \bigwedge^0 F \otimes \operatorname{Sym}_0 F \to 0 \to 0 \to \cdots,$$

which is not exact. If $p+q \neq 0$, then $L_{(p,1^q)F}$ is a syzygy of the complex (6.1.1); hence, we can describe it as a cokernel, as an image, and as a kernel:

$$\begin{tikzcd}
\cdots \arrow[r] & \bigwedge^p F \otimes \operatorname{Sym}_q F \arrow[rr] \arrow[dr] & & \bigwedge^{p-1} F \otimes \operatorname{Sym}_{q+1} F \arrow[r] & \cdots \\
& & L_{p,1^q}F \arrow[ur] & &
\end{tikzcd}$$

(d) $L_{f,\lambda}F = \bigwedge^f F \otimes L_\lambda F$.

Proposition 6.2. *If* $e_1, \dots, e_f$ *is a basis for the vector space* F *and* $\lambda = (\lambda_1, \dots, \lambda_s)$ *is a partition, then the set*

$$\{(e_{a_{1,1}} \wedge \ldots \wedge e_{a_{1,\lambda_1}}) \otimes (e_{a_{2,1}} \wedge \ldots \wedge e_{a_{2,\lambda_2}}) \otimes \cdots \otimes (e_{a_{s,1}} \wedge \ldots \wedge e_{a_{s,\lambda_s}})\}$$

is a basis for $L_\lambda(F)$ *where the rows of the following picture are strictly increasing and the columns are non-decreasing*:

$$\begin{array}{ccccccc} a_{1,1} & \cdots & a_{1,\lambda_s} & \cdots & a_{1,\lambda_2} & \cdots & a_{1,\lambda_1} \\ a_{2,1} & \cdots & a_{2,\lambda_s} & \cdots & a_{2,\lambda_2} & & \\ \vdots & & \vdots & & & & \\ a_{s,1} & \cdots & a_{s,\lambda_s} & & & & \end{array} \qquad (6.2.1)$$

Proof. First we show that the indicated elements generate $L_\lambda F$. **We deal with two rows.** We may arrange the indices in each row to be strictly increasing because each row corresponds to an element of $\bigwedge^\bullet F$. If some column is wrong, then we have

$$\begin{array}{ccccccccccc} a_1 & < & \cdots & < & \mathbf{a_w} & < & \cdots & < & \mathbf{a_{\lambda_2}} & < & \cdots \quad \mathbf{a}_{\lambda_1} \\ & & & & \vee & & & & & & \\ \mathbf{b_1} & < & \cdots & < & \mathbf{b_w} & < & \cdots & < & b_{\lambda_2} & & \end{array}$$

The image of

$$(a_1 \wedge \cdots \wedge a_{w-1}) \otimes (a_w \wedge \cdots a_{\lambda_1} \wedge b_1 \wedge \cdots \wedge b_w) \otimes (b_{w+1} \wedge \cdots \wedge b_{\lambda_2})$$

under

$$\bigwedge^{w-1} F \otimes \bigwedge^{\lambda_1-(w-1)+w} F \otimes \bigwedge^{\lambda_2-w} F \to \bigwedge^{\lambda_1} F \otimes \bigwedge^{\lambda_2} F \qquad {}^{3}$$

is ± our evil element plus tableaux which are lexicographically SMALLER than our evil tableaux. That is, these tableaux look like

$$\begin{matrix} a'_1 & < & \cdots & < & a'_w & < & \cdots & < & a'_{\lambda_2} & < & \cdots & a'_{\lambda_1} \\ b'_1 & < & \cdots & < & b'_w & < & \cdots & < & b'_{\lambda_2} \end{matrix}$$

The least index i with $a'_i \neq a_i$ has $a'_i < a_i$. (This least index comes on or before w.)

To deal with an arbitrary number of rows: Put a lexicographic order on all filled-in tableaux of shape λ. In other words, the tableaux (6.2.1) corresponds to the word:

$$a_{1,1}a_{1,2}\cdots a_{1,\lambda_1}a_{2,1}a_{2,2}\cdots a_{2,\lambda_2}\cdots\ \cdots\ \cdots a_{s,1}a_{s,2}\cdots a_{s,\lambda_s}$$

If $a'_1a'_2\cdots$ and $a_1a_2\cdots$ are words (with the same number of letters), then $a'_1a'_2\cdots$ is lexicographically less than $a_1a_2\cdots$ if the least index i with $a'_i \neq a_i$ has $a'_i < a_i$. Start with an arbitrary filled-in tableaux of shape λ. Apply the two-row process as many times as are needed until the original element has been written as a linear combination of elements from the proposed basis.

We use three steps to show that the proposed basis for $L_\lambda F$ is linearly independent.

Ultimately, we produce a homomorphism from $L_\lambda F$ to a vector space V with an ordered basis $\{b_i\}$ so that each element ξ of our proposed basis for $L_\lambda F$ is sent to

$$1\cdot b_{i(\xi)} + \sum_{i(\xi)+1\le j} \text{coefficient}_j\, b_j$$

and

$$\xi \neq \xi' \implies i(\xi) \neq i(\xi').$$

Step 1. We define a map

$$\phi_\lambda : \bigwedge^{\lambda_1} F \otimes \cdots \otimes \bigwedge^{\lambda_s} F \to \operatorname{Sym}_{\lambda'_1} F \otimes \cdots \otimes \operatorname{Sym}_{\lambda'_{s'}} F,$$

where $\lambda^{\text{transpose}} = (\lambda'_1, \dots, \lambda'_{s'})$. This map is called the Schur map associated to λ.

Step 2. We show that ϕ_λ factors through $L_\lambda F$.

Step 3. We show that ϕ_λ carries the proposed basis for $L_\lambda F$ to a linearly independent set in $\operatorname{Sym}_{\lambda'_1} F \otimes \cdots \otimes \operatorname{Sym}_{\lambda'_{s'}} F$.

Carry out Step 1. If $\lambda = (\lambda_1, \dots, \lambda_s)$ is a partition with $\lambda^{\text{transpose}}$ equal to $(\lambda'_1, \cdots, \lambda'_{s'})$, then define

$$\phi_\lambda : \bigwedge^{\lambda_1} F \otimes \bigwedge^{\lambda_2} F \otimes \cdots \otimes \bigwedge^{\lambda_s} F \to \operatorname{Sym}_{\lambda'_1} F \otimes \cdots \otimes \operatorname{Sym}_{\lambda'_{s'}} F$$

---

[3] Notice that "$u+v$" equals $w-1+\lambda_2-w = \lambda_2 - 1 < \lambda_2$.

to be the composition

$$\begin{matrix} \bigwedge^{\lambda_1} F \\ \otimes \\ \bigwedge^{\lambda_2} F \\ \otimes \\ \vdots \\ \otimes \\ \bigwedge^{\lambda_s} F \end{matrix} \xrightarrow{\begin{matrix}\Delta\\ \otimes \\ \vdots \\ \otimes \\ \Delta\end{matrix}} \begin{matrix} \underbrace{F\otimes\cdots\otimes F}_{\lambda_1} \\ \otimes \\ \vdots \\ \otimes \\ \underbrace{F\otimes\cdots\otimes F}_{\lambda_s}\end{matrix} \xrightarrow{\text{mult} \otimes \cdots \otimes \text{mult}} \operatorname{Sym}_{\lambda'_1} F \otimes \cdots \otimes \operatorname{Sym}_{\lambda'_{s'}} F$$

Carry out Step 2. Focus on part of a relation from row a and row $a+1$.

$$\begin{array}{cccccccccccc} x_1 & \cdots & x_u & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet \\ \bullet & \cdots & \bullet & \bullet & \cdots & \bullet & z_1 & \cdots & z_v & & & \end{array}$$

The black dot spaces are to be filled in with $y_1, \cdots, y_{\lambda_a - u + \lambda_{a+1} - v}$. The fact that $u + v < \lambda_{a+1}$ guarantees that at least one column has two dots. Pick the left most such column. Take two particular y's: $y^\dagger$ and $y^\ddagger$. $\underline{\text{For each choice of fill in}}$ of the $\underline{\underline{\text{rest}}}$ of the dots, there are two ways to arrange these two y's:

$$\begin{array}{ccccccccc} * & \cdots & * & y^\dagger & * & \cdots & * & \cdots & * \\ * & \cdots & * & y^\ddagger & * & \cdots & * & & \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} * & \cdots & * & y^\ddagger & * & \cdots & * & \cdots & * \\ * & \cdots & * & y^\dagger & * & \cdots & * & & \end{array}$$

These two tableaux come with opposite sign; hence their sum goes to zero under ϕ_λ. The defining submodule of $L_\lambda F$ is generated by such sums.

Carry out Step 3. Each basis element

$$e_{a_{1,1}} \cdots e_{a_{1,\mu_1}} \otimes e_{a_{2,1}} \cdots e_{a_{2,\mu_2}} \otimes \cdots \otimes e_{a_{t,1}} \cdots e_{a_{t,\mu_t}}$$

in $\operatorname{Sym}_{\mu_1} F \otimes \cdots \otimes \operatorname{Sym}_{\mu_t} F$, with $\mu_1 \geq \cdots \geq \mu_t$ and

$$a_{i,1} \leq \cdots \leq a_{i,\mu_i}$$

corresponds to a word

$$a_{1,1} \cdots a_{1,\mu_1} a_{2,1} \cdots a_{2,\mu_2} \cdots a_{t,1} \cdots a_{t,\mu_t}.$$

I can order these words lexicographically if I want. Now I have an ordered basis $\{b_1, \ldots, b_N\}$ for $\operatorname{Sym}_{\mu_1} F \otimes \cdots \otimes \operatorname{Sym}_{\mu_t} F$. Any set of elements

$$b_{i_1} + \text{h.o.t.}, \quad b_{i_2} + \text{h.o.t.}, \quad \cdots, \quad b_{i_\ell} + \text{h.o.t.}$$

in $\operatorname{Sym}_{\mu_1} F \otimes \cdots \otimes \operatorname{Sym}_{\mu_t} F$, with $i_1 < i_2 < \cdots < i_\ell$, is linearly independent.

Notice that ϕ_λ(proposed basis element 6.2.1) is

$$1 \cdot b_{\text{word read from column 1, column 2, ... column s}} + \text{h.o.t.} \ .$$

In other words, the least basis vector from $\operatorname{Sym}_{\lambda'_1} F \otimes \ldots \otimes \operatorname{Sym}_{\lambda'_{s'}} F$ which appears in ϕ_λ(proposed basis element (6.2.1)) has coefficient 1 and is not repeated as a least basis vector as (6.2.1) roams over all of the proposed basis for $L_\lambda F$. It follows that the proposed basis for $L_\lambda F$ is linearly independent. □

Bonus Comment. We showed that the Schur module $L_\lambda F$ is isomorphic to the image of the Schur map ϕ_λ.

6.B. **Sometimes Representation Theory alone gives a complex.** This discussion comes from a paper I wrote with Weyman.

Let F and G be vector spaces over the field $\boldsymbol{k}$ of dimension f and g respectively;

$$R = \operatorname{Sym}_{\boldsymbol{k}}(F \otimes G^*)$$

and $\phi : F \otimes_{\boldsymbol{k}} R \to G \otimes_{\boldsymbol{k}} R$ be the natural R-module homomorphism:

$$a \mapsto \sum_j g_j \otimes (a \otimes g_j^*),$$

for $a \in F$ and $g_1, \dots, g_g$, and $g_1^*, \dots, g_g^*$ a pair of dual bases for G and G^*. (As always, $\sum g_j \otimes g_j^*$ is a canonical element of $G \otimes G^*$.)

For each partition $\nu = (\nu_1, \dots, \nu_{g-1})$, I will give you a collection t_ν of free R-modules and R-module homomorphisms:

$$\cdots \to t_{\nu,k} \to t_{\nu,k-1} \to \dots .$$

Each t_ν is a complex, and is acyclic; see [7, Thm. 4.7]. It is easy to see that $H_0(t_\nu)$ is a module over $R/I_g(\phi)$. When we look at the complexes we will see that the length of t_ν is $f - g + 1$ when $\nu_1 \le f - g + 1$; so in these cases $H_0(t_\nu)$ is a maximal Cohen Macaulay $R/I_g(\phi)$-module (and a perfect R-module).

The notation. Given the partition $\nu = (\nu_1, \dots, \nu_{g-1})$ and an integer k.

- Find i with $\nu_i \ge k > \nu_{i+1}$. Let

$$p(\nu, k) = (\nu_1, \dots, \nu_i, k, \nu_{i+1} + 1, \dots, \nu_{g-1} + 1).$$

Let

$$N(\nu, k) = |p(\nu, k)| - |\nu| = k + g - 1 - i.$$

The modules. Let $t_{\nu,k} = \bigwedge^{N(\nu,k)} F \otimes_{\boldsymbol{k}} L_{p(\nu,k)'} G^* \otimes_{\boldsymbol{k}} R$. (Today I write λ' for $\lambda^{\text{transpose}}$.)

Example 6.3. • If $g = 4$, then $t_{(0,0,0)}$ is

$$\cdots \to \bigwedge^6 F \otimes \underbrace{L_{(3,1,1,1)'} G^*}_{D_2 G^* \otimes \bigwedge^4 G^*} \otimes R \to \bigwedge^5 F \otimes \underbrace{L_{(2,1,1,1)'} G^*}_{D_1 G^* \otimes \bigwedge^4 G^*} \otimes R$$

$$\to \bigwedge^4 F \otimes \underbrace{L_{(1,1,1,1)'} G^*}_{\bigwedge^4 G^*} \otimes R \to \bigwedge^0 F \otimes \underbrace{L_{(0,0,0,0)'} G^*}_{\boldsymbol{k}} \otimes R$$

(This is the Eagon-Northcott complex which resolves $R/I_4(\phi)$.)

- If $g = 4$, then $t_{(1,1,1)}$ is

$$\cdots \to \bigwedge^6 F \otimes \underbrace{L_{(3,2,2,2)'}G^*}_{D_1G^*\otimes\bigwedge^4 G^*\otimes\bigwedge^4 G^*} \otimes R \to \bigwedge^5 F \otimes \underbrace{L_{(2,2,2,2)'}G^*}_{\bigwedge^4 G^*\otimes\bigwedge^4 G^*} \otimes R$$

$$\to \bigwedge^1 F \otimes \underbrace{L_{(1,1,1,1)'}G^*}_{\bigwedge^4 G^*} \otimes R \to \bigwedge^0 F \otimes \underbrace{L_{(1,1,1,0)'}G^*}_{\mathrm{Sym}_1 G\otimes\bigwedge^4 G^*} \otimes R$$

(This is the "Buchsbaum-Rim" complex which resolves the cokernel of the generic map $F \xrightarrow{\phi} G$.)

- If $g = 4$, then $t_{(2,1,0)}$ is

$$\cdots \to \bigwedge^7 F \otimes L_{(4,3,2,1)'}G^* \otimes R \to \bigwedge^6 F \otimes L_{(3,3,2,1)'}G^* \otimes R \to \bigwedge^4 F \otimes L_{(2,2,2,1)'}G^* \otimes R$$

$$\to \bigwedge^2 F \otimes L_{(2,1,1,1)'}G^* \otimes R \to \bigwedge^0 F \otimes L_{(2,1,0,0)'}G^* \otimes R.$$

I included this example merely to point out that the family of complexes under consideration is much larger than the family of Eagon-Northcott complexes. One can tell the degree of the differential by looking at the difference in the power of $\bigwedge F$. The present example has three matrices of quadratic maps before linear maps finally appear. An Eagon-Northcott complex (see for example Appendix A2.6 in Eisenbud) has linear maps in every position except one.

The differentials. I will tell you the differential

$$t_{\nu,k} \to t_{\nu,k-1}. \tag{6.3.1}$$

The partition ν is $(\nu_1, \dots, \nu_i, \underbrace{k-1, \dots, k-1}_{m}, \nu_j, \dots, \nu_{g-1})$, with $\nu_i \geq k$ and $k-2 \geq \nu_j$. In this case,

$$p(\nu, k) = (\alpha, k^{m+1}, \beta) \quad \text{and} \quad p(\nu, k-1) = (\alpha, (k-1)^{m+1}, \beta)$$

for $\alpha = (\nu_1, \dots, \nu_i)$ and $\beta = (\nu_j + 1, \dots, \nu_{g-1} + 1)$. When $\boldsymbol{k}$ has characteristic zero, Representation Theory establishes the existence of a map

$$L_{(\alpha,k^{m+1},\beta)'}G^* \xrightarrow{\mathrm{RT}} L_{(1^{m+1})'}G^* \otimes L_{(\alpha,(k-1)^{m+1},\beta)'}G^*. \tag{6.3.2}$$

Of course, $L_{(1^{m+1})'}G^*$ is a very odd way of writing $\bigwedge^{m+1} G^*$. The map (6.3.1) is

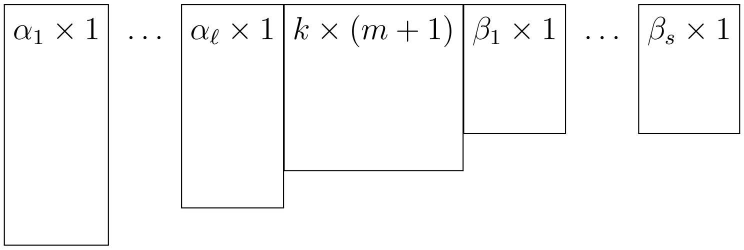
$$t_{\nu;k} = \bigwedge^{N(\nu;k)} F \otimes L_{(\alpha,k^{m+1},\beta)'}G^* \otimes R \xrightarrow{\mathrm{RT}} \bigwedge^{N(\nu;k)} F \otimes \bigwedge^{m+1} G^* \otimes L_{(\alpha,(k-1)^{m+1},\beta)'}G^*$$

$$\xrightarrow{\bigwedge^{m+1}\phi^*} \bigwedge^{N(\nu,k)} F \otimes \bigwedge^{m+1} F^* \otimes L_{(\alpha,(k-1)^{m+1},\beta)'}G^* \xrightarrow{\text{module action}}$$

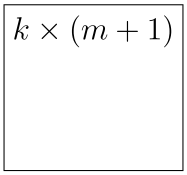
$$\bigwedge^{N(\nu;k)-(m+1)} F^* \otimes L_{(\alpha,(k-1)^{m+1},\beta)'}G^* = t_{\nu,k-1}$$

(In other words, use Representation Theory to pull $m+1$ boxes from row k and then do the "obvious map" involving the $m+1 \times m+1$ minors of ϕ.)

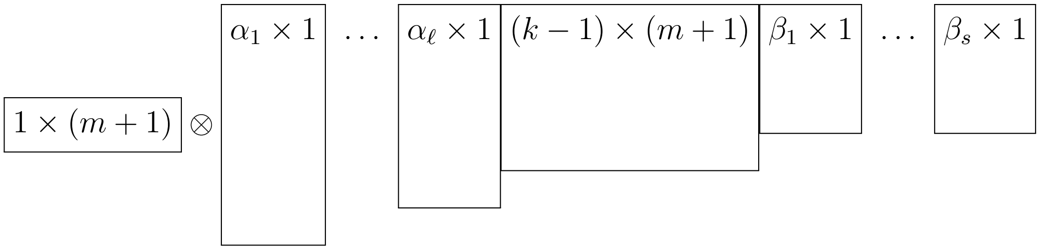
The picture that goes with (6.3.2) is:



The Representation Theory allows us to move the bottom row from



and get



The reason that Representation Theory gives the map (6.3.2): The module $L_{m+1}G^* \otimes L_{(\alpha,(k-1)^{m+1},\beta)'}G^*$ is a direct sum of irreducible representations of $\mathrm{GL}(G^*)$. The Littlewood-Richardson rule tells us that exactly one copy of $L_{(\alpha,k^{m+1},\beta)'}G^*$ appears in this direct sum decomposition. So (up to scalar multiple from $\boldsymbol{k}$), there is exactly one coordinate-free map (6.3.2). It is possible to write down exactly what (6.3.2) does, but it is not a very pretty answer!

6.C. **Each t_ν is a complex. The Littlewood-Richardson Rule** If λ and μ are partitions then

$$L_\lambda V \otimes L_\mu V = \bigoplus \mathrm{LR}(\lambda, \mu; \nu) L_\nu V$$

where the sum is taken over all partitions ν with $|\nu| = |\lambda| + |\mu|$, and the Littlewood-Richardson coefficient $\mathrm{LR}(\lambda, \mu; \nu)$ is is calculated according to the following description. Draw ν, remove μ. Fill in the resulting picture using λ_1 ones, λ_2 twos, etc. You must have your rows WEAKLY increasing and your columns STRICTLY increasing. The word that you form using the Macdonald convention (right to left top to bottom) must be a lattice permutation meaning $w = a_1 a_2 \dots a_N$ in the symbols $1, 2, \dots, n$ is a lattice permutation if for $1 \le r \le N$ and $1 \le i \le n-1$, the number of occurrences of the symbol i in $a_1 a_2 \dots a_r$ is not less than the number of occurrences of $i + 1$.

Example 6.4. Let us calculate the LR coefficient for $L_{(\alpha,k^{m+1},\beta)'}G^*$ in

$$L_{m+1}G^* \otimes L_{(\alpha,(k-1)^{m+1},\beta)'}G^*.$$

We draw $(\alpha, k^{m+1}, \beta)'$ and remove $(\alpha, (k-1)^{m+1}, \beta)'$. This leaves

$$\boxed{\text{a one } m+1 \text{ row of boxes}}.$$

We must fill these boxes in using $m+1$ ones. There is one way to do this. This unique way is weakly increasing in the rows and the word is okay! **Thus, there is exactly one non-zero $\mathrm{GL}(V)$-module homomorphism**

$$L_{(\alpha,k^{m+1},\beta)'}G^* \to L_{m+1}G^* \otimes L_{(\alpha,(k-1)^{m+1},\beta)'}G^*$$

(up to multiplication by a scalar).

Calculation. Now we show that each t_ν is a complex. That is, we show that the composition

$$t_{\nu,k} \to t_{\nu,k-1} \to t_{\nu,k-2} \tag{6.4.1}$$

is zero. Write $\nu = (\nu_1, \dots, \nu_i, (k-1)^a, (k-2)^b, \nu_j, \dots, \nu_{g-1})$, with $\nu_i \geq k$ and $k-3 \geq \nu_j$. Let $\alpha = (\nu_1, \dots, \nu_i)$ and $\beta = (\nu_j + 1, \dots, \nu_{g-1} + 1)$. We see that

$$p(\nu; k) = (\alpha, k^{a+1}, (k-1)^b, \beta), \quad p(\nu; k-1) = (\alpha, (k-1)^{a+b+1}, \beta)$$

and

$$p(\nu; k-2) = (\alpha, (k-1)^a, (k-2)^{b+1}, \beta).$$

The composition (6.4.1) is

$$t_{\nu,k} = \bigwedge^{N(\nu;k)} F \otimes L_{(\alpha,k^{a+1},(k-1)^b,\beta)'}G^* \otimes R \xrightarrow{\mathrm{RT}}$$

$$\bigwedge^{N(\nu;k)} F \otimes \bigwedge^{a+1} G^* \otimes L_{(\alpha,(k-1)^{a+1},(k-1)^b,\beta)'}G^* \otimes R \xrightarrow{\phi^*}$$

$$\bigwedge^{N(\nu;k)} F \otimes \bigwedge^{a+1} F^* \otimes L_{(\alpha,(k-1)^{a+1},(k-1)^b,\beta)'}G^* \otimes R \xrightarrow{\mathrm{MA}}$$

$$\bigwedge^{N(\nu;k)-(a+1)} F \otimes L_{(\alpha,(k-1)^{a+1},(k-1)^b,\beta)'}G^* \otimes R \xrightarrow{\mathrm{RT}}$$

$$\bigwedge^{N(\nu;k)-(a+1)} F \otimes \bigwedge^{b+1} G^* \otimes L_{(\alpha,(k-1)^a,(k-2)^{b+1},\beta)'}G^* \otimes R \xrightarrow{\phi^*}$$

$$\bigwedge^{N(\nu;k)-(a+1)} F \otimes \bigwedge^{b+1} F^* \otimes L_{(\alpha,(k-1)^a,(k-2)^{b+1},\beta)'}G^* \otimes R \xrightarrow{\mathrm{MA}}$$

$$\bigwedge^{N(\nu;k)-(a+1)-(b+1)} F \otimes L_{(\alpha,(k-1)^a,(k-2)^{b+1},\beta)'}G^* \otimes R = t_{\nu;k-2}$$

It is legal to do both representation theory maps first, then do both ϕ^* maps, and then do both module action maps. So we focus on the composition of the Representation Theory maps:

$$L_{(\alpha,k^{a+1},(k-1)^b,\beta)'}G^* \xrightarrow{\text{RT}} \bigwedge^{a+1} G^* \otimes L_{(\alpha,(k-1)^{a+1},(k-1)^b,\beta)'}G^* \xrightarrow{\text{RT}}$$

$$\bigwedge^{a+1} G^* \otimes \bigwedge^{b+1} G^* \otimes L_{(\alpha,(k-1)^a,(k-2)^{b+1},\beta)'}G^* \xrightarrow{\text{EM}} \bigwedge^{a+1+b+1} G^* \otimes L_{(\alpha,(k-1)^a,(k-2)^{b+1},\beta)'}G^*$$

(The maps "ϕ^*" and "Module action" both commute with "Exterior Multiplication".) Well, the Littlewood-Richardson rule tells us that the only $\mathrm{GL}(G^*)$-module map

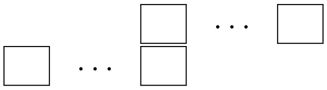
$$L_{(\alpha,k^{a+1},(k-1)^b,\beta)'}G^* \to L_{a+b+2}G^* \otimes L_{(\alpha,(k-1)^a,(k-2)^{b+1},\beta)'}G^* \tag{6.4.2}$$

is ZERO, and (6.4.1) factors through (6.4.2); thus, (6.4.1) is also zero.

To see that the only choice for a $\mathrm{GL}(G^*)$-module map (6.4.2) is zero: We compute the LR coefficient for $L_{(\alpha,k^{a+1},(k-1)^b,\beta)'}G^*$ in

$$L_{a+b+2}G^* \otimes L_{(\alpha,(k-1)^a,(k-2)^{b+1},\beta)'}G^*.$$

We draw the picture for $(\alpha, k^{a+1}, (k-1)^b, \beta)'$, remove the picture for $(\alpha, (k-1)^a, (k-2)^{b+1}, \beta)'$. We are left with



with $b+1$ boxes in the top row, $a+1$ boxes in the bottom row, and an overlap of one box. There is NO way to fill the picture in using ALL ONES so that the columns are strictly increasing. Thus, **the only coordinate free *k*-vector space map with domain and range given in (6.4.2) is zero** and (6.4.1) is also zero.

CONTENTS

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