Some problems, January 26, 2018

Let $R = \boldsymbol{k}[x, y]$, where $\boldsymbol{k}$ is a field, and I be the ideal $I = (x^3, x^2y, y^2)$ of R.

(a) Find a resolution of R/I by free R-modules. (You can make your resolution minimal and homogeneous if you feel so inclined.)

(b) Prove your answer to (a). (There are many options here. You could make up your own argument. You can use the argument I used when I gave the graduate colloquium in November. Problem (a) was an example in that talk. That lecture remains available. You can use the "What makes a complex exact?" Theorem. You can use the Hilbert-Burch Theorem.)

(c) Identify a regular sequence f_1, f_2 in I on R. (There are many answers here. We do not all have to have the same answer.)

(d) Notice that

$$\operatorname{Ext}^2_R(R/I, R) \cong \operatorname{Hom}_R(R/I, R/(f_1, f_2)) \cong \frac{(f_1, f_2) : I}{(f_1, f_2)}.$$

Recall that if A and B are ideals of a ring R, then $A : B$ is the ideal

$$\{r \in R \mid rB \subseteq A\}$$

of R.

(e) Find a resolution of $\frac{(f_1,f_2):I}{(f_1,f_2)}$ by free R-modules. (This part is easy if you have done the earlier parts. This part is the reason I made up these problems.)

(f) Give an explicit generating set for $(f_1, f_2) : I$.

(g) View $\frac{(f_1,f_2):I}{(f_1,f_2)}$ and R/I as graded vector spaces. It might be fun to see that $\frac{(f_1,f_2):I}{(f_1,f_2)}$ looks like R/I turned up-side-down.

(h) It might be fun to prove that $\frac{(f_1,f_2):I}{(f_1,f_2)}$ is an injective R/I-module. (You would have to look up "injective module".) You might even want to prove that $\frac{(f_1,f_2):I}{(f_1,f_2)}$ is the injective envelope of $\boldsymbol{k}$ as an R/I-module. Parts of this might be hard; the "easy" part is showing that $\frac{(f_1,f_2):I}{(f_1,f_2)}$ is an essential extension of $\boldsymbol{k}$ as an R/I-module.