Course:	Math 748
	Local Cohomology in Algebra, Geometry, and Topology
Semester:	Fall 2015
Instructor:	Andy Kustin
Potential audience:	Some knowledge of commutative algebra and/or algebraic geometry would be useful

Local cohomology provides a way to see things backwards. It is a valuable tool for computing Castelnuovo-Mumford regularity (how thick a projective resolution is)[1] and the arithmetic rank of an ideal.[2] It is the main tool in the proof of the Hartshorne-Lichtenbuam vanishing Theorem; it is also the main tool in the proof of theorems of Grothendieck and Hartshorne concerning the connectedness of various schemes, and it provides the proper framework for studying local duality.

Local cohomology modules are Artinian, but (in general) not finitely generated over the original ring; however local cohomology modules usually are finitely generated as D-modules, that is, as modules over the Weyl algebra of differential operators

$$W = \mathbb{C}[x_1, \dots, x_n, \tfrac{\partial}{\partial x_1}, \dots, \tfrac{\partial}{\partial x_n}].$$

The ring W is not completely commutative, none-the-less, it gives rise to a beautiful representation theory.

The course will be based on two sets of lecture notes:

- Notes of Craig Huneke:

  `{http://homepages.math.uic.edu/~bshipley/huneke.pdf}`

- Notes of Mel Hochster:

  `http://www.math.lsa.umich.edu/~hochster/615W11/loc.pdf}`

Other valuable references include

- The lectures given at "Local cohomology and its interactions with algebra, geometry, and analysis" from Snowbird in 2005:

  `{http://www.cmi.univ-mrs.fr/masters/master2mf/lib/exe/ fetch.php?media=local_cohomology_hypertext.pdf}`

- Peter Schenzel's paper "On The Use Of Local Cohomology In Algebra And Geometry":

  `{http://users.informatik.uni-halle.de/~schenzel/forsch/ lc0.pdf}`

- Tom Marley's lectures on local cohomology:

  `{http://digitalcommons.unl.edu/cgi/viewcontent.cgi ?article=1008&context=mathclass}`

NOTES

[1]By the way, here is Daniel Erman's answer on Math Overflow to the question "What is interesting/useful about Castelnuovo-Mumford regularity?":

Here's how I think about Castelnuovo-Mumford regularity. It's an invariant of an ideal (or module or sheaf) which provides a measure of how complicated that ideal (or module or sheaf) is. This invariant is related to free resolutions, and thus it measures complexity from that perspective.

Why is it interesting? One answer is that it can be used to provide an effective bound for two famous theorems. The first theorem I have in mind is that the Hilbert function of a graded ideal (or a finitely generated graded module) over the polynomial ring eventually agrees with the Hilbert polynomial of that ideal (or module). The second theorem I have in mind is Serre vanishing, which says that, given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, there exists d such that $\mathrm{H}^i(\mathbb{P}^n, \mathcal{F}(e)) = 0$ for all $i > 0$ and all $e > d$. These two theorems are related: if M is a graded module of depth > 0, and $\mathcal{F}$ is the associated sheaf of M, then the Hilbert function of M in degree e equals $\mathrm{H}^0(\mathbb{P}^n, \mathcal{F}(e))$.

An example where Castelnuovo-Mumford is particularly useful comes from the construction of the Hilbert scheme (I have heard that this is related to Mumford's original use, though I have no reference.) The basic point is that you can parametrize the set of ideals with a given Hilbert function by considering subloci of certain Grassmanians satisfying determinantal criteria, whereas it's less clear (at least to me) how to parametrize ideals with a given Hilbert polynomial.

Another great example where Castelnuovo-Mumford is useful is presented in Eisenbud "The Geometry of Syzygies" chapter 4, where he solves the interpolation problem for points in affine space.

[2]The *arithmetic rank* of an ideal I in a ring R is the least number of elements $f_1, \ldots, f_r$ in R such that I and the ideal $(f_1, \ldots, f_r)$ have the same radical. (If V is an algebraic variety, then the arithmetic rank of the ideal $I(V)$ is the least number of equations needed to define V. The variety V in projective n-space is called a *set-theoretic complete intersection* if the arithmetic rank of $I(V)$ is equal to n minus the dimension of V. It is a long-standing open question if every irreducible curve in complex projective 3-space ($\mathbb{P}^3_{\mathbb{C}}$) is a set-theoretic intersection of two surfaces in $\mathbb{P}^3_{\mathbb{C}}$. Local cohomology provides the best tool for obtaining a lower bound on the arithmetic rank of an ideal.