Homework 8

This problem is due not too much after the last day of class.

(27) Let R be the polynomial ring $k[x_1, x_2, x_3, x_4, y_1, y_2, y_3]$ and A be the ring R/I where I is the ideal generated by the 2×2 minors of

$$M = \begin{bmatrix} x_1 & x_2 & x_3 & y_1 & y_2 \\ x_2 & x_3 & x_4 & y_2 & y_3 \end{bmatrix}.$$

Let P be the ideal $(x_1, x_2, x_3, y_1, y_2)A$. Compute a generating set for $P^{(n)}$ for all n.

Remarks. These remarks might be interesting or they might be irrelevant; but you don't have to do anything with them.

(a) The ring A is a normal domain of Krull dimension three (i.e., the corresponding variety has dimension three). The ideal P of A is prime and A/P has Krull dimension two, which is ONE less than the Krull dimension of A. (So P is a "divisor" on A.)

(b) This problem is interesting because P generates the "divisor class group" of A. Consequently, every divisor on A is isomorphic to $P^{(n)}$ or $Q^{(n)}$, for some integer $n \geq 0$, where Q is the inverse of P in the divisor class group of A. (In this problem, $Q = (x_1, x_2)A$.) In particular, if $\mathfrak{P}$ is any prime ideal of A with the Krull dimension of $A/\mathfrak{P}$ equal to 2, then every every symbolic power of $\mathfrak{P}$, (including $\mathfrak{P}$ itself) is isomorphic to $P^{(n)}$ or $Q^{(n)}$, for some integer $n \geq 0$. (Furthermore, the isomorphism is given by multiplication by an element of the fraction field of A.)

(c) If the problem I asked is too hard, then only do some n, or make one or both of the scrolls shorter.

(d) If the problem I asked is too easy, then allow the scrolls to have arbitrary size and/or allow an arbitrary number of scrolls.

(e) You might be interested in describing the subvariety of $\mathbb{A}^7$ which is defined by I.