Homework 6

Due at the beginning of class on Wednesday, April 1.

(21) Problem 17 is an example of the present problem. Consider a 3×2 matrix M. The entries of M are homogeneous elements of $R = k[x_1, x_2]$. Each entry in the first column of M has degree d_1. Each entry in the second column of M has degree d_2. Let $\Delta_1, \Delta_2, \Delta_3$ be the three 2×2 minors of M. (Of course, these minors are homogeneous of degree $d = d_1 + d_2$.) Arrange these minors and give them signs so that

$$[\,\Delta_1 \quad \Delta_2 \quad \Delta_3\,]\, M = 0.$$

Let $\phi\colon \mathbb{A}^2 \to \mathbb{A}^3$ be the morphism

$$\phi(a,b) = (\Delta_1(a,b), \Delta_2(a,b), \Delta_3(a,b)).$$

Let $k[\mathbb{A}^3] = k[y_1, y_2, y_3]$. Identify two homogeneous polynomials F, G in $(k[y_1, y_2, y_3])[x_1, x_2]$ so that the resultant

$$\text{(*)} \qquad \operatorname{Res}(F, G) \text{ is in } I(\operatorname{im}\phi).$$

Prove (*).

(22) I want you to use Commutative Algebra or Algebraic Geometry to do problem 5.9 on page 88 in Hassett. Let S be the polynomial ring $k[x,y,z]$, and I be the ideal $I = (x^{m_1}, y^{m_2}, z^{m_3})$.

(a) Notice that

$$0 \xrightarrow{d_4} S(-m_1-m_2-m_3) \xrightarrow{d_3} \begin{matrix} S(-m_2-m_3) \\ \oplus \\ S(-m_1-m_3) \\ \oplus \\ S(-m_1-m_2) \end{matrix} \xrightarrow{d_2} \begin{matrix} S(-m_1) \\ \oplus \\ S(-m_2) \\ \oplus \\ S(-m_3) \end{matrix} \xrightarrow{d_1} S \xrightarrow{d_0} S/I \xrightarrow{d_{-1}} 0$$

is an exact sequence (that is $\ker d_i = \operatorname{im} d_{i+1}$ for all i), where d_0 is the natural quotient map, $d_1 = [\,x^{m_1} \quad y^{m_2} \quad z^{m_3}\,]$,

$$d_2 = \begin{bmatrix} 0 & z^{m_3} & -y^{m_2} \\ -z^{m_3} & 0 & x^{m_1} \\ y^{m_2} & -x^{m_1} & 0 \end{bmatrix}, \quad \text{and} \quad d_3 = \begin{bmatrix} x^{m_1} \\ y^{m_2} \\ z^{m_3} \end{bmatrix}.$$

(b) Let $S(-m)_d$ be the vector space of homogeneous polynomials in S of degree $d-m$. Notice that

$$\begin{aligned}\dim(S/I)_d = {} & \dim S_d - \Big(\dim S(-m_1)_d + \dim S(-m_2)_d + \dim S(-m_3)_d\Big) \\ & + \Big(\dim S(-m_2-m_3)_d + \dim S(-m_1-m_3)_d + \dim S(-m_1-m_2)_d\Big) \\ & - \dim S(-m_1-m_2-m_3)_d\end{aligned}$$

for all integers d.

(c) Identify d_0 so that $(S/I)_{d_0} \neq 0$, but $(S/I)_{d_0+1} = 0$. What is a basis for $(S/I)_{d_0}$?

(d) Apply (b) at d_0+1 to obtain the formula of problem 5.9.