**Seminar ideas, Spring 2014**
**written January, 17, 2014**

Basically, I propose two projects. Project one is self-contained and mainly mop-up duty. Project two is quite open-ended, but I will sub-divide it into self-contained pieces.

**Project 1.** I think there is still more to do to make "Tor" be a fully well-defined idea. One might want to prove that $\mathrm{Tor}^i(M,-)$ of $N$ is equal to $\mathrm{Tor}^i(-,N)$ of $M$ by showing that there are many ways to compute the homology of the Total complex of a double complex with exact rows and columns. (One can form the Total complex and then take the homology. One can look at the column of zeroth homology of the rows and compute the homology of this column. One can look at the row of zeroth homology of the columns and compute the homology of this row. It is not hard (i.e. ridiculously easy) to show that all three approaches give the same answer.) The advantage of this approach is that it also covers SCP-3 below and SCP-4 below. The disadvantage is that if you try to look it up you may well be sent to spectral sequences and I really have something much simpler in mind. So, feel free to get me to show you more of what I have in mind. Or, ignore Total complexes and just work out the exercise that I posted on the web in the fall (which I stole from somewhere (probably Rotman) a long time ago.)

**Project 2.** It would be fun to prove that various ideals are prime ideals. I outlined "my favorite" approach to this in class on Thursday, Jan. 16. It would be really cool to flesh out the details. I have two explicit ideals in mind. (Although, obviously, these ideals can easily be generalized.)

**The basic data.**

- Let $R_1$ be the polynomial ring $k[x_{1,1},x_{1,2},x_{2,1},x_{2,2},x_{3,1},x_{3,2}]$ in 6 variables over a field $k$ and let $I_1$ be the ideal of $R_1$ generated by the $2\times 2$ minors of the matrix
$$M_1 = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \end{bmatrix}.$$
- Let $R_2$ be the polynomial ring $k[x_{1,2},x_{1,3},x_{1,4},x_{1,5},x_{2,3},x_{2,4},x_{2,5},x_{3,4},x_{3,5},x_{4,5}]$ in 10 variables over a field $k$ and let $I_2$ be the ideal of $R_2$ generated by the maximal order Pfaffians of alternating matrix
$$M_2 = \begin{bmatrix} 0 & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ -x_{1,2} & 0 & x_{2,3} & x_{2,4} & x_{2,5} \\ -x_{1,3} & -x_{2,3} & 0 & x_{3,4} & x_{3,5} \\ -x_{1,4} & -x_{2,4} & -x_{3,4} & 0 & x_{4,5} \\ -x_{1,5} & -x_{2,5} & -x_{3,5} & -x_{4,5} & 0 \end{bmatrix}.$$
(For each index $i$ with $1\le i\le 5$, cross out row $i$ and column $i$ from $M_2$ and take the Pfaffian of the resulting $4\times 4$ alternating matrix. The ideal $I_2$ is generated by these

5 Pfaffians.) The Pfaffian of

$$\begin{bmatrix} 0 & x_{1,2} & x_{1,3} & x_{1,4} \\ -x_{1,2} & 0 & x_{2,3} & x_{2,4} \\ -x_{1,3} & -x_{2,3} & 0 & x_{3,4} \\ -x_{1,4} & -x_{2,4} & -x_{3,4} & 0 \end{bmatrix}$$

is $x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3}$. Notice that the square of the Pffafian of an alternating matrix is equal to the determinant of the matrix.

**Here are some of the specific goals of Project 2.**

**Specific Goal (1)** Prove $I_j$ is a prime ideal in $R_j$ for $j$ equals 1 or 2.

**Specific Goal (2)** Find the largest index $i$ so that the ring $R_j/I_j$ satisfies the Serre condition $(R_i)$ for each $j$.

**Specific Goal (3)** Prove that $R_j/I_j$ is Cohen-Macaulay for each $j$.

**Specific Goal (4)** Realize that (2) fits in a theoretical context!

**Specific Goal (5)** Find the minimal resolution of $R_j/I_j$ by free $R_j$ modules for each.

**Specific Goal (6)** Learn what Ext is.

**Specific Goal (7)** Learn how $\mathrm{Ext}^{\bullet}_A(M,N)$ measures the length of the longest regular sequence in the annihilator of M on N.

**Specific Goal (8)** Learn about linkage and how one can use linkage to create free resolutions.

**Here are some of the self-contained smaller projects that you can do to move us in the direction of accomplishing Project 2.**

**Self-Contained Project (1)** Identify a complex which is a CANDIDATE to be the minimal resolution of $R_j/I_j$ by free $R_j$-modules. (This is not hard!)

**Self-Contained Project (2)** Prove that the candidates of SCP(1) really are resolutions. (Once you have this, then you will be done with Specific Goal 4. Here you have (at least) two options.

- Option 1. Use the Buchsbaum-Eisenbud criteria as presented by Reid. (Remember that this criteria continues to work even if you did not hear Reid talk about it or even if you did not understand the proof. I usually don't understand a Theorem until I try to apply it.) It is very easy to show that the ranks work correctly. It probably isn't too hard to show the grade condition. (It is obvious that one can specialize (by specialize I mean set linear forms equal to zero) in such a way to get the grade condition. One "need only" argue that this is good enough. Surely, I published this argument many times; and of course other folks have also. You will have to look various theorems up about how the height of an ideal changes as you specialize; but that is why the Internet was invented. I switched from grade to height because the calculation takes place in the ambient ring $R_j$ which is Cohen-Macaulay. In a Cohen-Macaulay ring, the two concepts are the same!)

- Option 2. Find two very easy complexes and a map between your easy complexes so that the mapping cone of your map of complexes is the CANDIDATE of SCP(1). Apply SCP-3. (By the way, you are now doing linkage!) (The words I have written so far work for $R_1/I_1$. They have to be modified to work for $R_2/I_2$. Either, do what I wrote above twice, or use what Matt Miller and I called "Tight Double Linkage" (I think this technique is reprized in the Journal of Algebra paper with Vraciu and Rahmati.) Or – my real suggestion – make up your own plan.

**Self-Contained Project (3)** Figure out what the mapping cone of a map of complexes is and prove that there is a long exact sequence of homology which corresponds to a mapping cone. (Cameron and Tyler probably have done all of the heavy lifting here. You will not have to do it again.) This is an easy project – it is included in Lang's famous homework problem "open any book in homological algebra and …". If you want an example – look at option 2 in SCP 2.

**Self-Contained Project (4)** Look up the definitions of $\operatorname{Ext}_A^\bullet(-,M)$ and $\operatorname{Ext}_A^\bullet(N,-)$. Decide that these are meaningful functors. (Again Cameron and Tyler have done – or will have done – the heavy lifting here. If you are trying to come to grips with this Ext, you merely have to decide, "Ha, I use the snake lemma.") What is the long exact sequence of homology which corresponds to EACH of these functors? (This takes care of Specific Goal (6).)

**Self-Contained Project (5)** Figure out Lemma 1.2.4 on page 9 in Bruns and Herzog. This Lemma together with the nearby stuff DOES Specific goal 7. Ha!

**Self-Contained Project (6)** Now it is time to learn about perfect modules, grade, and projective dimension. I think Bruns and Vetter (Springer Lecture Notes in Mathematics 1327) pages 27–29 and/or 206–210 are perfect for this. Or Hochster's Topics in the homological theory of modules over commutative rings, proposition 6.14 on page 40 and near by stuff. The main Theorem is something like this: Let $I$ be an ideal in a Noetherian ring $A$, then the following statements hold.

(1) $\operatorname{grade} I \le \operatorname{pd}_A(A/I)$.
(2) If equality holds in (1), then $I$ is called a *perfect* ideal of $A$,
(3) If $I$ is a perfect ideal of a Cohen-Macaulay ring, then $A/I$ is a Cohen-Macaulay ring.
(4) Let $I$ be a perfect ideal of $A$, $F$ be a resolution of $A/I$ by free $A$-modules of length equal to $\operatorname{pd}_A(A/I)$, and $A \to B$ be a homomorphism of Noetherian rings. If $\operatorname{grade} I \le \operatorname{grade} IB$, then $\operatorname{grade} I = \operatorname{grade} IB$, $IB$ is a perfect ideal of $B$, and $B \otimes_A F$ is a resolution of $B/IB$ by free $B$-modules.
(5) If $I$ is a perfect ideal of a Cohen-Macaulay ring $A$ and $F$ is a resolution of $A/I$ by free $A$ with the length of $F$ equal to $\operatorname{pd}_A A/I$, then $\operatorname{Hom}_A(F,A)$ is a **resolution** of some module and that module is called *the canonical module* of $A/I$. (I added this part after I watched Jesse's lecture today.) Of course, this module is also called $\operatorname{Ext}_A^g(A/I,A)$.

When I write the *grade* of the ideal $I$ in the ring $A$ is $r$, I mean that the longest regular sequence in $I$ on $A$ has length $r$. The symbol pd stands for *projective dimension*.

This is a valuable theorem because if $A$ is say a polynomial ring over a field, then $A$ is Cohen-Macaulay (what ever that means) and now you have a homological sufficient condition ($\operatorname{grade} I = \operatorname{pd}_A(A/I)$) for showing that a quotient ring $A/I$ is Cohen-Macaulay. (If you restrict your attention to homogeneous ideals, or replace the polynomial ring by a localization of a polynomial ring, then this condition is necessary and sufficient.)

Do notice that SCP(2) and SCP(6) take care of Specific Goal (3).

**Self-Contained Project (7)** Lets deal with Specific Goal (2). Now you know the presentation matrix for $I_j$ and the height of $I_j$. Suppose $P$ is in $\operatorname{Spec} A_j/I_j$ and has small height, then $P$ misses lots of variables. Now you want to observe that some big piece of the presentation matrix of $I_j$ is invertible in $(A_j)_P$. (I guess $P$ is a prime of $A_j$ with $I_j \subseteq P$ and height $P(A_j/I_j)$ is small. Conclude that $(I_j)_P$ is generated by variables. Conclude that $(A_j/I_j)_P$ is isomorphic to a localization of $k[\text{the rest of the variables}]$; such a ring is regular (whatever regular means!) You have to take care of the numerology, but now you have a proof that $A_j/I_j$ satisfies the Serre condition $(R_i)$ for small $i$.

**Self-Contained Project (8)** It is time to finish Specific Goal (1). Srre proved that a ring is normal if and only if the ring satisfies $(R_1)$ and $(S_2)$. Your ring is Cohen-Macaulay which is much stronger than $(S_2)$. Your ring has satisfies $(R_1)$ so your ring is a normal ring. Prove that your ring has a connected spectrum.

**Self-Contained Project (9)** Focus on Option 2 in SCP(2). Look up "linkage". Observe that this option 2 is linkage. Do some more linkage, or not, as you like. This takes care of Specific Goal (8).

**Self-Contained Project (10)** Do Specific Goal (4). You might want to start by looking at Huneke and Ulrich, "The structure of Linkage", Annals. Look at page 278 and Theorem 4.2.