A STUDY OF SINGULARITIES ON RATIONAL CURVES BY WAY OF ROW IDEALS

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Introduction.

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The technique we use to separate the parameterizations which have base points or are non-birational from the parameterizations which are defined everywhere and are birational is roughly based on the idea of the Geometry of Syzygies (including Hilbert functions); see, for example Eisenbud's book [3] with the same title. Our separation of curves according to the number of visible singularities depends heavily on the "General Lemma" which originated in the work of Eisenbud and Ulrich [4] and has been exploited in [8]. The idea is that the generalized rows of the Hilbert-Burch matrix for a parameterization of a curve encode all of the information about the fibers of the parameterization. The finest separation which involves the sets

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C_i and counts nodes and cusps depends on calculations involving the module of Kähler differentials.

All of our work takes place over a field k. Often (but not always) the field is algebraically closed. Sometimes the field k has characteristic zero. Projective space $\mathbb{P}^n$ means projective space $\mathbb{P}^n_k$ over k. Let $\mathcal{O}_{\mathcal{C},p}$ represent the local ring of the curve $\mathcal{C}$ at the point p. The *singularity multiplicity* of p on $\mathcal{C}$, denoted m_p (or $m_{\mathcal{C},p}$ if there is any ambiguity about what $\mathcal{C}$ is), is the multiplicity of the local ring $\mathcal{O}_{\mathcal{C},p}$. (Recall that the multiplicity of the d-dimensional local ring $(A, \mathfrak{m})$ is $e(A) = \lim\limits_{n\to\infty} \frac{d!\lambda_A(A/\mathfrak{m}^n)}{n^d}$, where $\lambda_A(M)$ is the length of the A-module M.) The *number of branches* of $\mathcal{C}$ at p, denoted s_p (or $s_{\mathcal{C},p}$), is the number of minimal prime ideals of the completion $\widehat{\mathcal{O}_{\mathcal{C},p}}$ of $\mathcal{O}_{\mathcal{C},p}$ with respect to its maximal ideal $\mathfrak{m}_{\mathcal{C},p}$. The *singularity degree* of $\mathcal{C}$ at the point p, denoted δ_p (or $\delta_{\mathcal{C},p}$), is $\dim_k \overline{\mathcal{O}_{\mathcal{C},p}}/\mathcal{O}_{\mathcal{C},p}$, where $\overline{\mathcal{O}_{\mathcal{C},p}}$ is the normalization of $\mathcal{O}_{\mathcal{C},p}$. The invariant δ_p may also be realized as $\delta_p = \sum_q \binom{m_q}{2}$, where q varies over all singularities infinitely near to p. Let

$$\mathcal{C} = \mathcal{C}_0 \xleftarrow{\sigma_1} \mathcal{C}_1 \xleftarrow{\sigma_2} \dots \xleftarrow{\sigma_\ell} \mathcal{C}_\ell$$

be a sequence of blow-ups which desingularizes $\mathcal{C}$. The singular points on the curves $\mathcal{C}_i$, with $0 \le i \le \ell - 1$, which lie over p are the *singularities of $\mathcal{C}$ infinitely near to p*. When we write that the *multiplicity sequence* for the oscnode q_0 on the curve $\mathcal{C}_0$ is $(2 : 2 : 2 : 1, 1)$, we mean that there is a sequence of blow-ups:

$$\mathcal{C}_0 \xleftarrow{\sigma_1} \mathcal{C}_1 \xleftarrow{\sigma_2} \mathcal{C}_2 \xleftarrow{\sigma_3} \mathcal{C}_3$$

and a sequence of points q_0 on $\mathcal{C}_0$, q_1 on $\mathcal{C}_1$, q_2 on $\mathcal{C}_2$, and $q_3 \neq q_3'$ on $\mathcal{C}_3$, such that σ_i is the blow up of $\mathcal{C}_{i-1}$ centered at q_{i-1} for $1 \le i \le 3$,

$$\sigma_i^{-1}(q_{i-1}) = \begin{cases} q_i & \text{if } 1 \le i \le 2 \\ \{q_3, q_3'\} & \text{if } i = 3, \end{cases}$$

$$m_{\mathcal{C}_0,q_0} = m_{\mathcal{C}_1,q_1} = m_{\mathcal{C}_2,q_2} = 2 \quad \text{and} \quad m_{\mathcal{C}_3,q_3} = m_{\mathcal{C}_3,q_3'} = 1.$$

So, in particular,

$$\delta_{q_0} = \sum_{i=0}^{2} \binom{m_{q_i}}{2} = 3 \quad \text{and} \quad s_{q_0} = 2$$

because there are two smooth points on $\mathcal{C}_3$ which lie over q_0.

The following statements summarize our techniques. Theorem 0.1 follows from Lemma 1.7, which we call the General Lemma. This is a local result. Once one knows the singularities $\{p_i\}$ on a parameterized curve $\mathcal{C}$; then this result shows how to read $m_{\mathcal{C},p_i}$ and $s_{\mathcal{C},p_i}$, for each p_i, from the Hilbert-Burch matrix of the parameterization. Theorem 0.2 is a global result. It describes, in terms of the parameterization, all of the points p on $\mathcal{C}$ and all of the branches of $\mathcal{C}$ at p for which the multiplicity of p along the branch is at least two. The proof of Theorem 0.2 is carried out in Section 4 and involves studying the module of Kähler differentials. In contrast to Theorem 0.1, one may apply Theorem 0.2 before one knows the singularities on $\mathcal{C}$.

Cor1 **Theorem 0.1.** *Let k be an algebraically closed field and $\Psi\colon \mathbb{P}^1 \to \mathbb{P}^{n-1}$ be a morphism, with no base points, which is birational onto its image $\mathcal{C}$. Suppose that Ψ is given by $[g_1 : \cdots : g_n]$ for homogeneous polynomials $g_1, \dots, g_n$ of degree d in $k[x, y]$. Let φ be a Hilbert-Burch matrix for $[g_1, \dots, g_n]$, and $p_1, \dots, p_z$ be the visible singularities of $\mathcal{C}$. For each singular point p_j, let m_j be the singularity multiplicity of p_j on $\mathcal{C}$ and let s_j be the number of branches of $\mathcal{C}$ at p_j. Then the following statements hold:*

1. *The polynomial $\gcd I_1(p_j\varphi)$ in $k[x, y]$ has degree equal to m_j and has s_j distinct linear factors.*
2. *The polynomials $\gcd I_1(p_i\varphi)$ and $\gcd I_1(p_j\varphi)$ are relatively prime for $i \neq j$.*

Cor2' **Theorem 0.2.** *Retain the notation and hypotheses of Theorem 0.1. Assume in addition that the characteristic of k is zero. For each index j with $1 \le j \le z$, let $\gcd I_1(p_j\varphi) = \prod_{u=1}^{s_j} \ell_{u,j}^{e_{u,j}}$, where the $\ell_{u,j}$ are distinct linear factors and the exponents $e_{u,j}$ are positive. Let*

$$N = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \cdots & \frac{\partial g_n}{\partial x} \\ \frac{\partial g_1}{\partial y} & \cdots & \frac{\partial g_n}{\partial y} \end{bmatrix}$$

be the $2 \times n$ Jacobian matrix of the parametrization. Then

1. $\gcd I_2(N) = \prod_{j=1}^{z} \prod_{u=1}^{s_j} \ell_{u,j}^{e_{u,j}-1}$, *and*
2. *the degree of $\gcd I_2(N)$ is equal to $\sum_{j=1}^{z} (m_j - s_j)$.*

In Section 2 we ask when a set of polynomials has a common factor. We use the result of this section in our proof in Lemma 8.7 that the subsets S_i and C_i of $\mathbb{A}$ are closed.

If R is a ring, then we write $\operatorname{tqr}(R)$ for the *total quotient ring* of R; that is, $\operatorname{tqr}(R) = U^{-1}R$, where U is the set of non zerodivisors on R. If R is a domain, then the total quotient ring of R is usually called the *quotient field* of R and is denoted $\operatorname{qf}(R)$. The *normalization* $\overline{R}$ of a domain R is is the integral closure of R

in the qf(R). If $g_1, \dots, g_n$ are elements of $B = k[x,y]$ which generate an ideal of height two, then the Hilbert-Burch Theorem asserts that the relations on the row vector $[g_1, \dots, g_n]$ fit into an exact sequence

$$0 \to B^{n-1} \xrightarrow{\varphi} B^n \xrightarrow{[g_1,\dots,g_n]} B,$$

and that there is a unit u in B so that g_i is equal to $(-1)^{i+1}u$ times the determinant of φ with row i removed. We call φ a *Hilbert-Burch matrix* for $[g_1, \dots, g_n]$. When we factor a given polynomial f into powers of *distinct irreducible factors*: $f = \prod f_i^{e_i}$, we always mean that the irreducible factors f_i and f_j are not associates for $i \neq j$. If I and J are ideals of a ring R, then the *saturation of I with respect to J* is $I : J^\infty = \bigcup\limits_{i=1}^{\infty} (I : J^i)$. We write gcd to mean *greatest common divisor*. If I is a homogeneous ideal in $k[x,y]$, then the ideals $I : (x,y)^\infty$ and $(\gcd I)$ are equal. All rings in this paper are commutative and Noetherian. The expression "let $(A, \mathfrak{m}, k)$ be a local ring" means that A is a local ring with unique maximal ideal $\mathfrak{m}$ and k is the residue class field $A/\mathfrak{m}$. If M is a matrix, then $I_r(M)$ is the ideal generated by the $r \times r$ minors of M. If M is a graded module over a graded ring $B = B_0 \oplus B_1 \oplus \dots$, then the B_0-module M_d is the homogeneous component of M of degree d.

(1) **In the introduction we write that if $\boldsymbol{T} = [T_1, \dots, T_n]$ is a ANY SHAPE matrix of indeterminates and R is a ring, then $R[\boldsymbol{T}]$ is the polynomial ring $R[T_1, \dots, T_n]$.**
(2) define μ
(3) $\dim I = \dim R/I$ **in intro**
(4) Let M be a matrix with entries in a k-algebra, where k is a field. A *generalized row* of M is the product pM, a *generalized column* of M is the product Mq^{T}, and a generalized entry of M is the product pMq^{T}, where p and q are non-zero row vectors with entries from k. A *generalized zero* of M is a generalized entry of M which is zero.
(5) If M is a matrix, then M^{T} is the transpose of M.

Section 1. The General Lemma.

In this section we prove Lemma 1.7, which we call the General Lemma. Let $\mathcal{C}$ be the curve which is parameterized by the polynomials $g_1, \dots, g_n$ and let p in $\mathbb{P}^{n-1}$ be the point $(a_1 : \cdots : a_n)$ on $\mathcal{C}$. If φ is a Hilbert-Burch matrix for $[g_1, \dots, g_n]$, then the row vector $[a_1, \dots, a_n]\varphi$ captures a significant amount of geometric information about the behavior of $\mathcal{C}$ at p. Indeed, from this row vector, one may read the singularity multiplicity $m_{\mathcal{C},p}$ of $\mathcal{C}$ at p, the number of branches $s_{\mathcal{C},p}$ of $\mathcal{C}$ at p, and the multiplicity of p along each branch.

Lemma 1.1 is a preliminary result. One consequence of Lemma 1.1 is the well-known correspondence between the minimal prime ideals of the completion $\widehat{\mathcal{O}_{\mathcal{C},p}}$ of the local ring $\mathcal{O}_{\mathcal{C},p}$ and the maximal ideals of the integral closure $\overline{\mathcal{O}_{\mathcal{C},p}}$ of $\mathcal{O}_{\mathcal{C},p}$. The version we offer holds quite generally; our proof is an adaptation of the proof of [7, Thm. 16.14].

!32.1 **Lemma 1.1.** *Let T be the integral closure of the local one-dimensional domain $(R, \mathfrak{m})$. Suppose T is finitely generated as a module over R. Let $\hat{R}$ and $\hat{T}$ represent the completions of R and T in the $\mathfrak{m}$-adic topology. Then the the ring $\hat{R}$ is reduced and there is a one-to-one correspondence between the minimal prime ideals of $\hat{R}$ and the maximal ideals of T. If $\mathcal{M}$ is a maximal ideal of T, then the corresponding minimal prime ideal of $\hat{R}$ is $\ker(\hat{R} \to \hat{T}_{\mathcal{M}\hat{T}})$.*

Proof. We first establish the one-to-one correspondence between prime ideals. The ring $\hat{T}$ is finitely generated as an $\hat{R}$-module. The Cohen-Seidenberg Theorems guarantee that T and $\hat{T}$ are semi-local rings. Let $\mathfrak{M}_1, \dots, \mathfrak{M}_s$ be the maximal ideals of $\hat{T}$. Elementary properties of completion (see for example [9, Thm 8.15] or [2, Theorem 7.6]) ensure that

(a) the functions $\mathcal{M} \to \mathcal{M}\hat{T}$ and $\mathfrak{M} \cap T \leftarrow \mathfrak{M}$ give one-to-one correspondences between the set of maximal ideals of T and the set of maximal ideals of $\hat{T}$,
(b) the natural map $\hat{T} \to \hat{T}_{\mathfrak{M}_1} \times \cdots \times \hat{T}_{\mathfrak{M}_s}$ is an isomorphism, and
(c) the localization $\hat{T}_{\mathfrak{M}_i}$ of the complete ring $\hat{T}$ at its maximal ideal $\mathfrak{M}_i$ is equal to the completion $\widehat{T_{\mathfrak{M}_i \cap T}}$ of the local ring $T_{\mathfrak{M}_i \cap T}$ at its maximal ideal $(\mathfrak{M}_i \cap T)T_{\mathfrak{M}_i \cap T}$.

The ring T is a one-dimensional normal domain; so, T satisfies the Serre conditions (R_1) and (S_2). In particular, the local ring $T_{\mathfrak{M}_i \cap T}$ is a Discrete Valuation Ring (DVR) and the completion of $T_{\mathfrak{M}_i \cap T}$, which is equal to $\hat{T}_{\mathfrak{M}_i}$, is a complete DVR. The ring extension $\hat{R} \to \hat{T}$ is an integral extension; so every prime ideal of $\hat{R}$ has the form $P \cap \hat{R}$, where P is a prime ideal in $\hat{T}$. We know $\operatorname{Spec} \hat{T}$ completely. This spectrum consists of distinct maximal ideals $\mathfrak{M}_1, \dots, \mathfrak{M}_s$ (each of which has height one) and distinct minimal ideals $\mathfrak{n}_1, \dots, \mathfrak{n}_s$, where $\mathfrak{n}_i \subseteq \mathfrak{M}_i$, and $\mathfrak{n}_i$ equal to the kernel of the localization map $\hat{T} \to \hat{T}_{\mathfrak{M}_i}$. All of the maximal ideals of $\hat{T}$ contract to the unique maximal ideal $\mathfrak{m}\hat{R}$ of $\hat{R}$. The minimal prime ideal $\mathfrak{n}_i$ of $\hat{T}$ contracts to a minimal prime ideal $\mathfrak{n}_i \cap \hat{R}$ of $\hat{R}$. It is clear that

$$\mathfrak{n}_i \cap \hat{R} = \ker\left(\hat{R} \xrightarrow{\text{inclusion}} \hat{T} \xrightarrow{\text{localization}} \hat{T}_{\mathfrak{M}_i}\right).$$

We need to establish that $\mathfrak{n}_1 \cap \hat{R}, \dots, \mathfrak{n}_s \cap \hat{R}$ are distinct.

To that aim, we compute the total quotient ring of $\hat{T}$ two different ways. First of all, the ring $\hat{T}$ is a direct product of DVRs; and therefore, the set of zero divisors of $\hat{T}$ is equal to the union of the minimal prime ideals of $\hat{T}$; hence,

!tqr1 (1.2)
$$\operatorname{tqr}(\hat{T}) \cong \operatorname{qf}(\hat{T}_{\mathfrak{M}_1}) \times \cdots \times \operatorname{qf}(\hat{T}_{\mathfrak{M}_s}).$$

On the other hand, Lemma 1.4 shows that $\hat{R}$ and $\hat{T}$ have the same total quotient ring. Let $J_1, \dots, J_u$ be the minimal prime ideals of $\hat{R}$. Every non-zero element of $\mathfrak{m}$ is an element of $\mathfrak{m}\hat{R}$ and is regular on $\hat{R}$. It follows that $\mathfrak{m}\hat{R}$ is not an associated prime ideal of $\hat{R}$. The only prime ideals of $\hat{R}$ are $\mathfrak{m}\hat{R}$ and $J_1, \dots, J_u$; thus, the complete set of zero divisors on $\hat{R}$ is $\cup J_i$. Every prime ideal of $\operatorname{tqr}(\hat{R})$ is a maximal ideal; so the ordinary Chinese Remainder Theorem gives that the natural map

$$\operatorname{tqr}(\hat{R}) \to \operatorname{tqr}(\hat{R})/J_1 \times \cdots \times \operatorname{tqr}(\hat{R})/J_u$$

is an isomorphism. It is not difficult to see that, for each i, the localization map $\operatorname{tqr}(\hat{R})/J_i \to \hat{R}_{J_i}/J_i$ is an isomorphism. Thus,

!tqr2 (1.3)
$$\operatorname{tqr}(\hat{R}) \cong \hat{R}_{J_1}/J_1 \times \cdots \times \hat{R}_{J_1}/J_u.$$

Compare (1.2) and (1.3). The ring $\operatorname{tqr}(\hat{T}) \cong \operatorname{tqr}(\hat{R})$ has exactly s maximal ideals and also has exactly u maximal ideals. We conclude that $s = u$ and the proof of (1) is complete.

We prove that R is analytically unramified (that is $\hat{R}$ is reduced) by showing that the local domain $(R, \mathfrak{m})$ satisfies condition (D) of [13, Chapt. VIII, sect. 13, Lemma 1]. That is, we show that there exists a non-zero element d of R with $dA \subseteq \hat{R}$, where A is the integral closure of $\hat{R}$ in $\operatorname{tqr}(\hat{R})$. In our situation, $A = \hat{T}$. Indeed, $\hat{T}$ is an integral extension of $\hat{R}$ with

$$\hat{R} \subseteq \hat{T} \subseteq \operatorname{tqr}(\hat{T}) = \operatorname{tqr}(\hat{R}),$$

and $\hat{T}$ is a normal ring. We know from Lemma 1.4 that there exists a non-zero element d in R with $d\hat{T} \subseteq \hat{R}$. $\square$

!.p171 **Lemma 1.4.** *Let $R \subseteq S \subseteq \operatorname{qf}(R)$ be rings with $(R, \mathfrak{m})$ a local domain and S a finitely generated R-module. Let $\hat{R}$ and $\hat{S}$ be the completions of R and S in the $\mathfrak{m}$-adic topology. Then $\hat{R}$ and $\hat{S}$ have the same total quotient ring and there exists a non-zero element d in R with $d\hat{S} \subseteq \hat{R}$.*

Proof. Let $U = R \setminus \{0\}$. Notice first that $\operatorname{qf}(R) = U^{-1}R = U^{-1}S$. Every element of U is regular on both rings $\hat{R} \subseteq \hat{S}$; so, $U^{-1}(\hat{R}) \subseteq U^{-1}(\hat{S})$. These two rings are equal because

$$\begin{aligned} U^{-1}(\hat{S}) = U^{-1}(S \otimes_R \hat{R}) &= U^{-1}(S) \otimes_{U^{-1}(R)} U^{-1}(\hat{R}) = U^{-1}(R) \otimes_{U^{-1}(R)} U^{-1}(\hat{R}) \\ &= U^{-1}(\hat{R}). \end{aligned}$$

We have $U^{-1}\hat{S} = U^{-1}\hat{R} \subseteq \operatorname{tqr} \hat{R} \subseteq \operatorname{tqr} \hat{S}$. A typical element of $\operatorname{tqr} \hat{S}$ is z/w, where z and w are in $\hat{S}$ with w regular on $\hat{S}$. There exists $u \in U$ with uz, uw in $\hat{R}$. Of course uw is regular on $\hat{R} \subseteq \hat{S}$. So, $z/w = uz/uw \in \operatorname{tqr} \hat{R}$. The first assertion is established. The second assertion follows from the fact that $\hat{S}$ is a finitely generated $\hat{R}$-module and $\hat{S} \subseteq U^{-1}\hat{R}$. $\square$

34.1 **Data 1.5.** Let k be a field, $g_1, \dots, g_n$ be homogeneous forms of degree d in the polynomial ring $B = k[x, y]$, $\Psi \colon \mathbb{P}^1 \to \mathbb{P}^{n-1}$ be the morphism which is given by $[g_1 : \dots : g_n]$, $\mathcal{C}$ be the image of Ψ, and I be the ideal $(g_1, \dots, g_n)B$ of B. Assume that

(1) I is minimally generated by $g_1, \dots, g_n$,
(2) I has height two, and
(3) $\Psi \colon \mathbb{P}^1 \to \mathcal{C}$ is a birational morphism.

Remark. The hypotheses imposed on the parameterization Ψ in Data 1.5 are fairly mild. Furthermore, if a given parameterization of a rational curve $\mathcal{C}$ fails to satisfy these hypotheses, one can reparameterize and obtain a parameterization of $\mathcal{C}$ which does satisfy the hypotheses. Hypothesis (1) is equivalent to the statement that "$\mathcal{C}$ is not contained in any hyperplane section of $\mathbb{P}^{n-1}$". Hypothesis (2) is equivalent to the statement "the morphism Ψ has no base points". The homogeneous coordinate ring of $\mathcal{C}$ is $k[g_1, \dots, g_n] = k[I_d]$ and the homomorphism $k[T_1, \dots, T_n] \twoheadrightarrow k[I_d]$, which sends T_i to g_i for each i, induces the isomorphism

Ts (1.6)
$$\frac{k[T_1, \dots, T_n]}{I(\mathcal{C})} \cong k[I_d],$$

where $I(\mathcal{C})$ is the ideal generated by the homogeneous polynomials which vanish on $\mathcal{C}$. The homogeneous coordinate ring for the Veronese curve of degree d is $k[x^d, x^{d-1}y, \dots, y^d] = k[B_d]$. Hypothesis (3) is equivalent to the statement "the domains $k[I_d] \subseteq k[B_d]$ have the same quotient field".

.gl **Lemma 1.7. (The General Lemma).** *Adopt Data 1.5. Fix the point p on $\mathcal{C}$. Assume that the fiber $\Psi^{-1}(p)$ is equal to the fiber $\Psi_{\bar{k}}^{-1}(p)$, where $\Psi_{\bar{k}}\colon \mathbb{P}^1_{\bar{k}} \to \mathbb{P}^{n-1}_{\bar{k}}$ is the extension of Ψ to a morphism over the algebraic closure $\bar{k}$ of k. Let $q_1, \dots, q_s$ be the s distinct points in $\mathbb{P}^1$ which comprise the fiber $\Psi^{-1}(p)$; $\mathfrak{p}$ be the prime ideal in $k[I_d]$ which corresponds to the point p on $\mathcal{C}$; $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ be the prime ideals in B which correspond to the points $q_1, \dots, q_s$ in $\mathbb{P}^1$, φ be a Hilbert-Burch matrix for $[g_1, \dots, g_n]$, and Δ be the greatest common divisor of the entries of the row vector $p\varphi$. Write $\Delta = \ell_1^{c_1} \cdots \ell_t^{c_t}$, where $\ell_1, \dots, \ell_s$ are pairwise non-associate irreducible homogeneous forms in B. Let $R \subseteq S$ be the rings $k[I_d]_{\mathfrak{p}} \subseteq k[B]_{\mathfrak{p}}$, $\hat{R} \subseteq \hat{S}$ be the completions of $R \subseteq S$ in the $\mathfrak{m}_R$-adic topology, $J_1, \dots, J_u$ be the minimal prime ideals of $\hat{R}$, and $\mathfrak{M}_1, \dots \mathfrak{M}_v$ be the maximal ideals of $\hat{S}$. Then $s = t = u = v = s_{\mathcal{C},p}$ and after re-numbering*

$$q_i = V(\ell_i), \quad \mathfrak{q}_i = \ell_i B, \quad \mathfrak{M}_i = \mathfrak{q}_i \hat{S}, \quad J_i = \ker(\hat{R} \to \hat{S}_{\mathfrak{M}_i}), \quad \textit{and} \quad c_i = e(\hat{R}/J_i),$$

for $1 \le i \le s$. In particular, $\deg \Delta = e(R) = e(\mathcal{O}_{\mathcal{C},p}) = m_{\mathcal{C},p}$, and $c_1, \dots, c_s$ are the multiplicities of p along the branches of $\mathcal{C}$ at p.

many **Remarks 1.8.** (a) We use the symbol p to represent a point $(a_1 : \dots : a_n)$ in $\mathbb{P}^{n-1}$ as well as a row vector $[a_1, \dots, a_n]$. The meaning will be clear from context.

(b) The important conclusions are $s = t$, $\mathfrak{q}_i = \ell_i B$, and $c_i = e(\hat{R}/J_i)$. The other statements are well-known, or follow easily from these, as we explain below.

(c) The prime ideal in $k[T_1, \dots, T_n]$ which corresponds to the point p in $\mathbb{P}^{n-1}$ is $I_2(M)$ for

$$M = \begin{pmatrix} a_1 & \dots & a_n \\ T_1 & \dots & T_n \end{pmatrix}.$$

The prime ideal $\mathfrak{p}$ in $k[I_d]$ which corresponds to p on $\mathcal{C}$ is $I_2(M)k[I_d]$, see (1.6); thus,

$$\mathfrak{p} = I_2 \begin{pmatrix} a_1 & \dots & a_n \\ g_1 & \dots & g_n \end{pmatrix} k[I_d].$$

(d) Our notation $S = B_{\mathfrak{p}}$ means that S is the localization $U^{-1}B$ of B at the multiplicatively closed set $U = k[I_d] \setminus \mathfrak{p}$. Let $T = k[B_d]_{\mathfrak{p}}$. (In other words, T is equal to $U^{-1}(k[B_d])$.) The ring inclusions $k[I_d] \subseteq k[B_d] \subseteq B$ are integral extensions; so the ring inclusions $R \subseteq T \subseteq S$ are integral extensions. The Veronese ring $k[B_d]$ is a normal domain and the domains $k[I_d] \subseteq k[B_d]$ have the same quotient field by hypothesis (3); hence, $k[B_d]$ is the normalization of $k[I_d]$ and T is the normalization of R.

(e) The ring inclusions $R \subseteq T \subseteq S$ are module finite extensions and R is a local ring with maximal ideal $\mathfrak{m}_R = \mathfrak{p}k[I_d]_{\mathfrak{p}}$. It follows from the Cohen-Seidenberg Theorems that T and S are semi-local rings. Moreover, $\operatorname{Proj} k[B_d] = \operatorname{Proj} B$; so the function $M \mapsto M \cap T$ gives a one-to-one correspondence between the maximal ideals of S and the maximal ideals of T.

(f) The $\mathfrak{m}_R$-adic topology on S is equivalent to the J-adic topology on S, where J is the Jacobson radical of S. It is well known (see, for example [9, Thm. 8.15] or [7, Thm. K.11]) that the natural map

$$\hat{S} \to \hat{S}_{\mathfrak{M}_1} \times \cdots \times \hat{S}_{\mathfrak{M}_v}$$

is an isomorphism; furthermore, the local ring $\hat{S}_{\mathfrak{M}_i}$ is complete for each i and $\hat{S}_{\mathfrak{M}_i}$ is equal to the completion of the local ring $S_{\mathfrak{M}_i \cap S}$ in the $(\mathfrak{M}_i \cap S)$-adic topology. Each ring $S_{\mathfrak{M}_i \cap S}$ is a one-dimensional regular ring; hence, each $\hat{S}_{\mathfrak{M}_i}$ is a complete DVR. Furthermore, the maximal ideals of S are $\{\mathfrak{M}_i \cap S \mid 1 \le i \le v\}$.

(g) The statements of Remark (f) also apply to T. So there is a commutative diagram

$$\begin{tikzcd}
\hat{S} \arrow[r, phantom, "="] & \hat{S}_{\mathfrak{M}_1} \arrow[r, phantom, "\times"] & \cdots \arrow[r, phantom, "\times"] & \hat{S}_{\mathfrak{M}_v} \\
\hat{T} \arrow[u, hook] \arrow[r, phantom, "="] & \hat{T}_{\mathfrak{M}_1 \cap \hat{T}} \arrow[u, hook] \arrow[r, phantom, "\times"] & \cdots \arrow[r, phantom, "\times"] & \hat{T}_{\mathfrak{M}_v \cap \hat{T}}, \arrow[u, hook]
\end{tikzcd}$$

where $\hat{T}_{\mathfrak{M}_i \cap \hat{T}}$ is the localization of the ring $\hat{T}$ at the maximal ideal $\mathfrak{M}_i \cap \hat{T}$ and also is the completion of the local ring $T_{\mathfrak{M}_i \cap T}$ at the maximal ideal $\mathfrak{M}_i \cap T$. Each map $\hat{T}_{\mathfrak{M}_i \cap \hat{T}} \hookrightarrow \hat{S}_{\mathfrak{M}_i}$ is an integral extension.

(h) Lemma 1.1 may be applied to the rings $R \subseteq T$ in order to see that the ring $\hat{R}$ is reduced, $u = v$, and, after renumbering, $J_i = \ker(\hat{R} \to \hat{T}_{\mathfrak{M}_i \cap \hat{T}})$. We enlarge the commutative diagram of (g) to obtain the commutative diagram:

$$\begin{tikzcd}
\hat{S} \arrow[r, phantom, "="] & \hat{S}_{\mathfrak{M}_1} \arrow[r, phantom, "\times"] & \cdots \arrow[r, phantom, "\times"] & \hat{S}_{\mathfrak{M}_v} \\
\hat{T} \arrow[u, hook] \arrow[r, phantom, "="] & \hat{T}_{\mathfrak{M}_1 \cap \hat{T}} \arrow[u, hook] \arrow[r, phantom, "\times"] & \cdots \arrow[r, phantom, "\times"] & \hat{T}_{\mathfrak{M}_v \cap \hat{T}} \arrow[u, hook] \\
\hat{R} \arrow[u, hook] \arrow[r, hook] & \frac{\hat{R}}{J_1} \arrow[u, hook] \arrow[r, phantom, "\times"] & \cdots \arrow[r, phantom, "\times"] & \frac{\hat{R}}{J_v}. \arrow[u, hook]
\end{tikzcd}$$

For each i, the ring $\hat{T}_{\mathfrak{M}_i \cap \hat{T}}$ is the normalization of $\frac{\hat{R}}{J_i}$.

Proof. The ring extension $k[I_d] \subseteq B$ is integral; so every maximal ideal of S has the form $\mathfrak{q}S$, where $\mathfrak{q}$ is a height one homogeneous prime ideal of B which is minimal over $\mathfrak{p}B$ and which satisfies $\mathfrak{q} \cap k[I_d] = \mathfrak{p}$. Let $\mathfrak{q}$ be a prime ideal in B for which $\mathfrak{q}S$ is a maximal ideal of S. The ideal $\mathfrak{q}$ is principal and is generated by some homogeneous form $f \in B$. Let $q \in \mathbb{P}^1_{\bar{k}}$ be a root of f. The generators of $\mathfrak{p}$, which may be found in Remark (c), are in the ideal $\mathfrak{q} = (f)$; and therefore, the generators of $\mathfrak{p}$ all vanish at q. It follows that $\Psi_{\bar{k}}(q) = p$. The hypothesis $\Psi_{\bar{k}}^{-1}(p) = \Psi^{-1}(p)$ ensures that q is already in $\mathbb{P}^1$; and therefore, $q \in \{q_1, \dots, q_s\}$, f is a linear polynomial, and $\mathfrak{q} \in \{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}$. We conclude that

maxs (1.9) $$\text{the maximal ideals of } S \text{ are } \{\mathfrak{q}_1 S, \dots, \mathfrak{q}_s S\}.$$

It follows from Remark (f) that $v = s$ and the maximal ideals of $\hat{S}$ are $\mathfrak{M}_i = \mathfrak{q}_i \hat{S}$, for $1 \le i \le s$.

We next show that

.rowop (1.10) $$\mathfrak{p}B : I = I_1(p\varphi).$$

To do this, we perform a matrix manipulation which produces a new generating set $g'_1, \dots, g'_n$ for I with the property that when g'_{i_0} is removed from this generating set, for some i_0, the remaining polynomials generate the ideal $\mathfrak{p}B$. The ideal $\mathfrak{p}B : I$ may then be read from the Hilbert-Burch matrix for the new generators of I.

The point p is in $\mathbb{P}^{n-1}$; so $a_{i_0} \neq 0$, for some i_0. Use Remark (c) to see that $\mathfrak{p}$ is generated by the $n-1$ polynomials

.gi0 (1.11) $$g'_j = \det \begin{bmatrix} a_{i_0} & a_j \\ g_{i_0} & g_j \end{bmatrix}, \text{ for } 1 \le j \le n \text{ with } j \neq i_0.$$

Let $g'_{i_0} = a_{i_0} g_{i_0}$. Let u_1 and u_2 be column vectors with n entries: the entry in position i_0 of u_1 is 1, all other entries are zero; and $u_2 = p^{\mathrm{T}} - a_{i_0} u_1$. Let M and $M\check{}$ be the $n \times n$ matrices $M = a_{i_0} I - u_1 u_2^{\mathrm{T}}$ and $M\check{} = a_{i_0} I + u_1 u_2^{\mathrm{T}}$. Observe that

$$[\, g_1, \dots, g_n \,]\, M = [\, g'_1, \dots, g'_n \,].$$

The vectors u_1 and u_2 are orthogonal to one another, so $M M\check{} = a_{i_0}^2 I$; so, $M\check{}\varphi$ is a Hilbert-Burch matrix for $[\, g'_1, \dots, g'_n \,]$. The left side of (1.10) is equal to the ideal generated by the entries of row i_0 of $M\check{}\varphi$; and this row is equal to $p\varphi$. We have established (1.10).

We compute the saturation $\mathfrak{p}B : (x,y)^\infty$ two different ways. On the one hand, $\mathfrak{p}B : (x,y)^\infty$ is equal to the intersection of the $\mathfrak{q}$-primary components of $\mathfrak{p}B$ as $\mathfrak{q}$ roams over all of the height one prime ideals of B in $\operatorname{Ass} B/\mathfrak{p}B$. For each such

$\mathfrak{q}$, the $\mathfrak{q}$-primary component of $\mathfrak{p}B$ is $\mathfrak{p}B_{\mathfrak{q}} \cap B$ and the ring $B_{\mathfrak{q}}$ is a DVR; so, $\mathfrak{p}B_{\mathfrak{q}} = \mathfrak{q}^w B_{\mathfrak{q}}$ for some exponent w. Thus,

$$\mathfrak{p}B\colon (x,y)^\infty = \mathfrak{q}_1^{(w_1)} \cap \cdots \cap \mathfrak{q}_s^{(w_s)} = \mathfrak{q}_1^{w_1} \cap \cdots \cap \mathfrak{q}_s^{w_s} = \mathfrak{q}_1^{w_1} \cdots \mathfrak{q}_s^{w_s}.$$

We have taken advantage of the fact that each $\mathfrak{q}_i$ is principal in the Unique Factorization Domain B. On the other hand, the ideal I is (x,y)-primary and ":" is associative, so we use (1.10) to see that

$$\begin{aligned}\mathfrak{p}B\colon (x,y)^\infty &= \mathfrak{p}B\colon I^\infty = (\mathfrak{p}B\colon I)\colon I^\infty = I_1(p\varphi)\colon I^\infty = I_1(p\varphi)\colon (x,y)^\infty = \Delta B\\ &= \ell_1^{c_1} \cdots \ell_t^{c_t} B.\end{aligned}$$

We have two factorizations of Δ into non-associate irreducible factors. We conclude that $s = t$, and after renumbering, $\ell_i B = \mathfrak{q}_i$ and $w_i = c_i$. It follows that each ℓ_i is a linear form and $q_i = V(\ell_i)$.

We next calculate the multiplicity $e(\hat{R}/J_i)$ for each i. Recall that the maximal ideals of T are

$$(\mathfrak{M}_i \cap \hat{T}) \cap T = (\mathfrak{q}_i \cap k[B_d])T \quad \text{for } 1 \le i \le s$$

and that the completion $(T_{(\mathfrak{q}_i \cap k[B_d])T})\widehat{\ }$ of the local ring $T_{(\mathfrak{q}_i \cap k[B_d])T}$ at the maximal ideal $(\mathfrak{q}_i \cap k[B_d])T_{(\mathfrak{q}_i \cap k[B_d])T}$ is equal to the localization $\hat{T}_{\mathfrak{M}_i \cap \hat{T}}$ of the complete ring $\hat{T}$ at the maximal ideal $\mathfrak{M}_i \cap \hat{T}$. We simplify the notation by letting

$$R_i = \hat{R}/J_i,\ T_i = (T_{(\mathfrak{q}_i \cap k[B_d])T})\widehat{\ } = \hat{T}_{\mathfrak{M}_i \cap \hat{T}}, \text{ and } \mathfrak{m}_{T_i} = (\mathfrak{q}_i \cap k[B_d])T_i = (\mathfrak{M}_i \cap \hat{T})T_i.$$

Recall from Remark (h) that T_i is the integral closure $\bar{R}_i$ of R_i. The maximal ideal of R_i is

$$\mathfrak{m}_{R_i} = \mathfrak{m}_{\hat{R}/J_i} = \mathfrak{m}_{\hat{R}}/J_i = (\mathfrak{m}_R \hat{R})/J_i = ((\mathfrak{p}R)\hat{R})/J_i = (\mathfrak{p}\hat{R})/J_i.$$

We know (see Remark (h)) that $J_i = \ker(\hat{R} \to T_i)$; so,

.mrt (1.12) $$\mathfrak{m}_{R_i} T_i = \mathfrak{p}T_i.$$

The value of $\mathfrak{p}T_i$ is completely determined by the $(\mathfrak{q}_i \cap k[B_d])$-primary component of $\mathfrak{p}k[B_d]$. We have already computed the primary components of $\mathfrak{p}B$ corresponding to the prime ideals minimal in the support of $B/\mathfrak{p}B$:

$$\mathfrak{p}B\colon (x,y)^\infty = \mathfrak{q}_1^{c_1} \cap \cdots \cap \mathfrak{q}_s^{c_s}.$$

The rings $k[B_d]_{\mathfrak{q}_i \cap k[B_d]} \subseteq B_{\mathfrak{q}_i}$ are DVRs. One can choose the same generator for the maximal ideal of these two rings. There is no difficulty in seeing that if f is an element of $k[B_d]$ and r is arbitrary, then

$$f \in (\mathfrak{q}_i \cap k[B_d])^r k[B_d]_{\mathfrak{q}_i \cap k[B_d]} \iff f \in \mathfrak{q}_i^r B_{\mathfrak{q}_i};$$

and therefore, there is no difficulty in seeing that the $(\mathfrak{q}_i \cap k[B_d])$-primary component of $\mathfrak{p}k[B_d]$ is $(\mathfrak{q}_i \cap k[B_d])^{c_i}$. It follows from (1.12) that

$$\mathfrak{m}_{R_i} T_i = \mathfrak{p} T_i = \mathfrak{m}_{T_i}^{c_i} T_i.$$

On the other hand, Observation 1.21 shows that $\mathfrak{m}_{R_i} T_i = \mathfrak{m}_{T_i}^{e(R_i)} T_i$ because $R_i \subseteq T_i$ are local one-dimensional domains with common residue class field $k(g_{i_0}) \subseteq R_i$, for g_{i_0} as found in (1.11); furthermore, T_i is the normalization of R_i and T_i is finitely generated as an R_i-module. Thus,

$$\mathfrak{m}_{T_i}^{c_i} T_i = \mathfrak{m}_{R_i} T_i = \mathfrak{m}_{T_i}^{e(R_i)} T_i,$$

and $c_i = e(R_i)$.

Next, we relate the degree of the polynomial Δ to the multiplicity of the local ring R. The modules $R/\mathfrak{m}_R^r$ and $\hat{R}/\mathfrak{m}_R^r \hat{R}$ are equal for all r; so $e(R) = e(\hat{R})$. The associativity formula for multiplicities yields

$$e(\hat{R}) = \sum_{i=1}^{s} e(\hat{R}/J_i).$$

Thus,

$$e(R) = e(\hat{R}) = \sum_{i=1}^{s} e(\hat{R}/J_i) = \sum_{i=1}^{s} c_i = \deg(\ell_1^{c_1} \cdots \ell_s^{c_s}) = \deg \Delta.$$

We translate the information we have collected about the rings $R \subseteq T$ and $\hat{R} \subseteq \hat{T}$ to information about the rings $\mathcal{O}_{\mathcal{C},p} \subseteq \overline{\mathcal{O}_{\mathcal{C},p}}$ and $\widehat{\mathcal{O}_{\mathcal{C},p}} \subseteq \left(\overline{\mathcal{O}_{\mathcal{C},p}}\right)^{\widehat{}}$. Recall first that all four rings are subrings of $\mathrm{qf}(k[I_d]) = \mathrm{qf}(k[B_d])$:

$$\begin{aligned}
\mathcal{O}_{\mathcal{C},p} &= \{\tfrac{f}{g} \mid \ f \in k[I_d] \text{ and } g \in k[I_d] \setminus \mathfrak{p} \text{ are homogeneous of the same degree}\},\\
\overline{\mathcal{O}_{\mathcal{C},p}} &= \{\tfrac{f}{g} \mid \ f \in k[B_d] \text{ and } g \in k[I_d] \setminus \mathfrak{p} \text{ are homogeneous of the same degree}\},\\
R = k[I_d]_{\mathfrak{p}} &= \{\tfrac{f}{g} \mid \ f \in k[I_d] \text{ and } g \in k[I_d] \setminus \mathfrak{p}\}, \text{ and}
\end{aligned}$$

$$T = k[I_d]_{\mathfrak{p}} = \{\tfrac{f}{g} \mid \ f \in k[B_d] \text{ and } g \in k[I_d] \setminus \mathfrak{p}\}.$$

Observe that g_{i_0} is a unit of R which is transcendental over $\mathcal{O}_{\mathcal{C},p}$. Recall that if $(A, \mathfrak{m})$ is a local ring and z is an indeterminate over A, then $A(z)$ means the ring $A[z]_{\mathfrak{m}A[z]}$. It is not difficult to check that the two subrings $\mathcal{O}_{\mathcal{C},p}(g_{i_0})$ and R of $\mathrm{qf}(k[I_d])$ are equal and therefore, R and $\mathcal{O}_{\mathcal{C},p}$ have the same multiplicity. The ring $\overline{\mathcal{O}_{\mathcal{C},p}}$ is a subring of T of R. Dehomogenization and homogenization

hd (1.13)
$$\begin{array}{rcl} \mathfrak{a} & \to & \mathfrak{a}T, \\ \mathfrak{A} \cap \overline{\mathcal{O}_{\mathcal{C},p}} & \leftarrow & \mathfrak{A}, \end{array}$$

provide a one-to-one correspondence between the maximal ideals $\mathfrak{a}$ of $\overline{\mathcal{O}_{\mathcal{C},p}}$ and the maximal ideals of $\mathfrak{A} \in \{\mathfrak{M}_i \cap T \mid 1 \le i \le v\}$ of T.

Finally, we compare the multiplicities $e(\hat{R}/J_i)$ and $e(\widehat{\mathcal{O}_{\mathcal{C},p}}/\mathcal{J}_i)$, where $\mathcal{J}_i$ is the minimal prime of $\widehat{\mathcal{O}_{\mathcal{C},p}}$ which corresponds to the minimal prime ideal J_i of $\hat{R}$. In this discussion, we use a generating set $\gamma_1, \dots, \gamma_n$ for I with the property that $\gamma_1, \dots, \gamma_{n-1}$ generate $\mathfrak{p}$ and $\gamma_n \notin \mathfrak{p}$. (Such a set may be obtained by renaming the generating set $g_1', \dots, g_n'$ of (1.11).) Define ring homomorphisms $\rho\colon k[t_1, \dots, t_{n-1}] \to \mathcal{O}_{\mathcal{C},p}$ and $\rho\colon k[t_1, \dots, t_n] \to R$ by setting

set (1.14)
$$\rho(t_i) = \gamma_i/\gamma_n \text{ for } 1 \le i \le n-1, \text{ and } \rho(t_n) = t_n.$$

big Observe that the homomorphisms ρ induce surjections:

$$\rho\colon k[t_1, \dots, t_{n-1}]_{(t_1,\dots,t_{n-1})} \twoheadrightarrow \mathcal{O}_{\mathcal{C},p} \tag{1.15}$$
$$\rho\colon k(t_n)[t_1, \dots, t_{n-1}]_{(t_1,\dots,t_{n-1})} \twoheadrightarrow R. \tag{1.16}$$

Let K be the kernel of (1.15). Observe that the kernel of (1.16) is equal to the extension of K to $k(t_n)[t_1, \dots, t_{n-1}]_{(t_1,\dots,t_{n-1})}$.

Focus on one maximal ideal $\mathfrak{N} = ((\ell)B \cap k[B_d])T$ of T, where $\ell = \ell_i$ for one fixed i. Let m be a linear form in B with ℓ, m a basis for B_1, and let $\mathfrak{N}^c$ represent the contraction $\mathfrak{N} \cap \overline{\mathcal{O}_{\mathcal{C},p}}$ of $\mathfrak{N}$ to $\overline{\mathcal{O}_{\mathcal{C},p}}$. There is no difficulty checking that $\left(\overline{\mathcal{O}_{\mathcal{C},p}}\right)_{\mathfrak{N}^c} = k[\frac{\ell}{m}]_{(\frac{\ell}{m})}$ and $T_{\mathfrak{N}} = k(\gamma_n)[\frac{\ell}{m}]_{(\frac{\ell}{m})}$. We have the following commutative diagram of rings and ring homomorphisms:

$$\begin{array}{ccc}
k[t_1,\ldots,t_{n-1}]_{(t_1,\ldots,t_{n-1})} & \subseteq & k(t_n)[t_1,\ldots,t_{n-1}]_{(t_1,\ldots,t_{n-1})} \\
\downarrow & & \downarrow \\
\frac{k[t_1,\ldots,t_{n-1}]_{(t_1,\ldots,t_{n-1})}}{Kk[t_1,\ldots,t_{n-1}]_{(t_1,\ldots,t_{n-1})}} & \subseteq & \frac{k(t_n)[t_1,\ldots,t_{n-1}]_{(t_1,\ldots,t_{n-1})}}{Kk(t_n)[t_1,\ldots,t_{n-1}]_{(t_1,\ldots,t_{n-1})}} \\
\| & & \| \\
\mathcal{O}_{\mathcal{C},p} & \subseteq & R \\
\hookdownarrow & & \hookdownarrow \\
\overline{\mathcal{O}_{\mathcal{C},p}} & \subseteq & T \\
\hookdownarrow & & \hookdownarrow \\
(\overline{\mathcal{O}_{\mathcal{C},p}})_{\mathfrak{N}^c} & \subseteq & T_{\mathfrak{N}} \\
\| & & \| \\
k[\frac{\ell}{m}]_{(\frac{\ell}{m})} & \subseteq & k(\gamma_n)[\frac{\ell}{m}]_{(\frac{\ell}{m})}.
\end{array}$$

Complete each local ring $(A,\mathfrak{m})$ in the above diagram in its $\mathfrak{m}$-adic topology to obtain the following commutative diagram of complete local rings

$$\begin{array}{ccc}
k[[t_1,\ldots,t_{n-1}]] & \subseteq & k(t_n)[[t_1,\ldots,t_{n-1}]] \\
\downarrow & & \downarrow \\
\frac{k[[t_1,\ldots,t_{n-1}]]}{Kk[[t_1,\ldots,t_{n-1}]]} & \subseteq & \frac{k(t_n)[[t_1,\ldots,t_{n-1}]]}{Kk(t_n)[[t_1,\ldots,t_{n-1}]]} \\
\| & & \| \\
\widehat{\mathcal{O}_{\mathcal{C},p}} & \subseteq & \hat{R} \\
\downarrow & & \downarrow \\
\left((\overline{\mathcal{O}_{\mathcal{C},p}})_{\mathfrak{N}^c}\right)\widehat{\ } & \subseteq & \widehat{T_{\mathfrak{N}}} = T_i \\
\| & & \| \\
k[[\frac{\ell}{m}]] & \subseteq & k(\gamma_n)[[\frac{\ell}{m}]].
\end{array}$$

big2 The compositions

$$k[[t_1,\dots,t_{n-1}]] \to k[[\tfrac{\ell}{m}]] \tag{1.17}$$

$$k(t_n)[[t_1,\dots,t_{n-1}]] \to k(\gamma_n)[[\tfrac{\ell}{m}]], \tag{1.18}$$

from top to bottom, are still given by ρ as defined in (1.14). Let $\mathfrak{j}$ be the kernel of (1.17) and $\mathfrak{J}$ be the kernel of (1.18). The ideal J_i is defined to be the kernel of $\hat{R} \to \widehat{T_{\mathfrak{N}}} = T_i$. The corresponding minimal prime ideal $\mathcal{J}_i$ of $\widehat{\mathcal{O}_{\mathcal{C},p}}$ is the kernel of $\widehat{\mathcal{O}_{\mathcal{C},p}} \to \big((\overline{\mathcal{O}_{\mathcal{C},p}})_{\mathfrak{N}^c}\big)\widehat{\ }$. We have

$$\begin{aligned} \mathcal{J}_i &= \mathfrak{j}/Kk[[t_1,\dots,t_{n-1}]], & J_i &= \mathfrak{J}/Kk(t_n)[[t_1,\dots,t_{n-1}]],\\ \widehat{\mathcal{O}_{\mathcal{C},p}}/\mathcal{J}_i &= k[[t_1,\dots,t_{n-1}]]/\mathfrak{j}, \text{ and} & \hat{R}/J_i &= k(t_n)[[t_1,\dots,t_{n-1}]]/\mathfrak{J}. \end{aligned}$$

Observe that $\mathfrak{j}$ may be thought of as the k-vector space of solutions to a system of linear homogeneous equations (L) over k. That is, $\rho(\sum A_a t_1^{a_1}\cdots t_{n-1}^{a_{n-1}})$ is equal to zero if and only if the coefficient of $(\frac{\ell}{m})^i$ in $\sum A_a \rho(t_1)^{a_1}\cdots\rho(t_{n-1})^{a_{n-1}}$ is zero for all i. The A_a are the unknowns. There is one equation for each power $(\frac{\ell}{m})^i$. The coefficients are in k. The field extension $k \subseteq k(\gamma_n)$ is flat; so, a k-basis for the solution set of (L) over k is also a $k(\gamma_n)$-basis for the solution set of (L) over $k(\gamma_n)$. We conclude that $\mathfrak{J} = \mathfrak{j}k(t_n)[[t_1,\dots,t_{n-1}]]$. The local rings

$$(k[[t_1,\dots,t_{n-1}]]/\mathfrak{j},\mathfrak{m}) \quad\text{and}\quad (k(t_n)[[t_1,\dots,t_{n-1}]]/\mathfrak{j}k(t_n)[[t_1,\dots,t_{n-1}]],\mathfrak{M})$$

have the same multiplicity because the $k(\gamma_n)$-vector spaces

$$(\mathfrak{m}^i/\mathfrak{m}^{i+1}) \otimes_k k(\gamma_n) \quad\text{and}\quad \mathfrak{M}^i/\mathfrak{M}^{i+1}$$

are equal for all i. Thus,

shw
$$e(\hat{R}/J_i) = e(\widehat{\mathcal{O}_{\mathcal{C},p}}/\mathcal{J}_i) \tag{1.19}$$

for all i. $\square$

ext **Remark 1.20.** Lemma 1.7 continues to hold (but is not very interesting), even if p is a point of $\mathbb{P}^{n-1}$ which is not on $\mathcal{C}$. Indeed, in this case, $\deg\Delta$ and $m_{\mathcal{C},p}$ are both zero because a quick look at Remark 1.8 (c) shows that $\mathfrak{p}B$ is an (x,y)-primary ideal; so (1.10) gives $(x,y)^n \subseteq \mathfrak{p}B : I = I_1(p\varphi) \subseteq (\Delta)$, for some n; so Δ must be a unit.

Proof of Theorem 0.1. Assertion (1) is explicitly stated as part of Lemma 1.7. We prove (2). Lemma 1.7 shows that $\gcd I_1(p_j\varphi) = \prod_{u=1}^{s_j} \ell_{u,j}^{e_{u,j}}$, where the linear factors $\ell_{u,j}$ correspond to the points in the fiber $\Psi^{-1}(p_j)$. If $i \neq j$, then the fibers $\Psi^{-1}(p_i)$ and $\Psi^{-1}(p_j)$ are disjoint, so; the polynomials $\gcd I_1(p_i\varphi)$ and $\gcd I_1(p_j\varphi)$ are relatively prime. $\square$

.O33.10' **Observation 1.21.** *Let $(A, \mathfrak{m}_A, k_A) \subseteq (B, \mathfrak{m}_B, k_B)$ be one-dimensional local domains. Assume that k_A is infinite. If B is the normalization of A and B is finitely generated as an A-module, then $\mathfrak{m}_A B = \mathfrak{m}_B^{e/r}$, where $e = e(A)$ is the multiplicity of the local ring A and $r = \lambda_A(k_B)$.*

Proof. The hypothesis about k_A ensures that there exists a minimal reduction $z \in \mathfrak{m}_A$ of $\mathfrak{m}_A$. The domain B is normal, local, and one dimensional; so, B is a Principal Ideal Domain and the equation $z\mathfrak{m}_A^u B = \mathfrak{m}_A^{u+1} B$, for some u, tells us that $zB = \mathfrak{m}_A B$. We compute

str
$$e = \lambda_A(A/zA) = \lambda_A(B/zB) = \lambda_B(B/zB)r. \tag{1.22}$$

The middle equality is obtained from the picture

$$\begin{tikzcd}
 & B \arrow[dl, dash] \arrow[d, dash] \\
A \arrow[d, dash] & zB \arrow[dl, dash] \\
zA &
\end{tikzcd}$$

All lengths are finite; in particular, $\lambda_A(B/A)$ is finite because A is a one dimensional domain and B is a module-finite extension of A with $B \subseteq \mathrm{qf}(A)$. Multiplication by z gives an isomorphism of A-modules $B/A \cong zB/zA$; therefore the A-modules A/zA and B/zB have the same length. The equality on the right is due to the fact that every factor in a composition series for the B-module B/zB is k_B. The formulas of (1.22) have been established and $\lambda_B(B/zB) = e/r$. The only quotient of B with length e/r, as a B-module, is $B/\mathfrak{m}_B^{e/r}$. We conclude that $\mathfrak{m}_A B = zB = \mathfrak{m}_B^{e/r}$. $\square$

Cor2 **Corollary 1.23.** *Let k be an algebraically closed field and $\Psi \colon \mathbb{P}^1 \to \mathbb{P}^{n-1}$ be a morphism, with no base points, which is birational onto its image $\mathcal{C}$. Suppose that Ψ is given by $[g_1 : g_2 : g_3]$ for homogeneous polynomials g_1, g_2, g_3 of degree d in $R = k[x, y]$ and that the Hilbert-Burch matrix for $[g_1, g_2, g_3]$ is*

$$0 \to R(-d-d_1) \oplus R(-d-d_2) \xrightarrow{\varphi} R(-d)^3 \xrightarrow{[g_1, g_2, g_3]} R,$$

where $d_1 + d_2 = d$ and $1 \le d_1 \le d_2$. Then $\mathcal{C}$ has a visible singularity of multiplicity $d - 1$ if and only if $d_1 = 1$.

Proof. ($\Rightarrow$) If p is a point on $\mathcal{C}$ with $m_p = d - 1$, then Lemma 1.7 shows that

$$d - 1 = \deg(\gcd I_1(p\varphi)) \le d_2 \le d - 1.$$

The left most inequality is due to the fact that the non-zero entries of the row vector $p\varphi$ have degree d_1 and/or d_2.

($\Leftarrow$) If $d_1 = 1$, then the entries of column 1 of φ come from a two-dimensional vector space; so, there exists a point p in $\mathbb{P}^2$ so that the left entry of $p\varphi$ is zero. The ideal (g_1, g_2, g_3) is 3-generated; so the ideal $I_1(p\varphi)$ is non-zero and is generated by a homogeneous form of degree $d-1$. Lemma 1.7 (together with Remark 1.20) show that p is a singularity on $\mathcal{C}$ of multiplicity $d-1$. $\square$

We close this section with the observation that every parameterization of a curve leads to a parameterization of the branches of the curve. Recall that if p is a point on a curve $\mathcal{C}$, then the minimal prime ideals of $\widehat{\mathcal{O}_{\mathcal{C},p}}$ are called the *branches* of $\mathcal{C}$ at p; furthermore, if $\mathcal{J}$ is a minimal prime of $\widehat{\mathcal{O}_{\mathcal{C},p}}$, then the multiplicity of the local ring $\widehat{\mathcal{O}_{\mathcal{C},p}}/\mathcal{J}$ is called the *multiplicity of p along the branch $\mathcal{J}$ of $\mathcal{C}$.*

.O34.2 **Observation 1.24.** *Adopt the Data of 1.5, with k algebraically closed. Then there is a one-to-one correspondence between the points of $\mathbb{P}^1$ and the branches of $\mathcal{C}$.*

Proof. Fix be a point q in $\mathbb{P}^1$. Let p be the point $\Psi(q)$ on $\mathcal{C}$. Form the ideal $\mathfrak{p}$ of $k[I_d]$ as described in Remark 1.8 (c) and form the rings

$$R = k[I_d]_{\mathfrak{p}} \subseteq T = k[B_d]_{\mathfrak{p}} \subseteq S = B_{\mathfrak{p}}$$

as described in Remark 1.8 (d). There are explicit, well-defined, one-to-one corre-
c5 spondences between each of the following sets:

$$\begin{tikzcd}
\Psi^{-1}(p) \arrow[r, leftrightarrow, "(1.25)"'] & \text{Max-Spec}(S) \arrow[r, leftrightarrow, "(1.26)"'] & \text{Max-Spec}(T) \arrow[d, leftrightarrow, "(1.27)"'] \\
\text{The branches of } \mathcal{C} \text{ at } p \arrow[r, leftrightarrow, "(1.29)"'] & \text{The Min Primes of } \widehat{\mathcal{O}_{\mathcal{C},p}} \arrow[r, leftrightarrow, "(1.28)"'] & \text{Max-Spec}(\overline{\mathcal{O}_{\mathcal{C},p}})
\end{tikzcd}$$

If q_1 is a point in $\Psi^{-1}(p)$ and $\mathfrak{q}_1$ is the homogeneous prime ideal of B which corresponds to q,then the correspondence (1.25) sends q_1 to $\mathfrak{q}S$ as shown in (1.9). The correspondence (1.26) is described in Remark 1.8 (e). The correspondence (1.27) is explained in (1.13). Lemma 1.1 accounts for (1.28), and (1.29) is the definition of branch. $\square$

brn **Remark 1.30.** We say that an ideal of $B = k[x, y]$ is a *linear ideal* if it is generated by one non-zero linear form. There is a one-to-one correspondence between the linear ideals of B and the points of $\mathbb{P}^1$. Thus, Observation 1.24 gives a one-to-one correspondence between the linear ideals of B and the branches of $\mathcal{C}$. If (ℓ) is a linear ideal of B, then let $\mathcal{C}(\ell)$ be the corresponding branch of $\mathcal{C}$. It makes sense to speak about the *multiplicity along the branch* $\mathcal{C}(\ell)$ because ℓ corresponds to a point q in $\mathbb{P}^1$, $\Psi(q) = p$ is a point on $\mathcal{C}$, $\mathcal{C}(\ell)$ is a minimal prime ideal of $\widehat{\mathcal{O}_{\mathcal{C},p}}$, and the multiplicity of the local ring $\widehat{\mathcal{O}_{\mathcal{C},p}}/\mathcal{C}(\ell)$ is called the multiplicity of p along the branch $\mathcal{C}(\ell)$ of $\mathcal{C}$.

Something like the next result should appear as a Corollary to the General Lemma.

ctgl **Corollary 1.31.** *Let $\mathcal{C}$ be a parameterized plane curve with a nice parameterization $\boldsymbol{g}$, φ be a homogeneous Hilbert Burch matrix for $\boldsymbol{g}$, (d_1, d_2) be the degree sequence for φ with $d_1 \le d_2$, and p be a point on $\mathcal{C}$. The following statements hold.*

(1) *The singularity multiplicity $m_p \le d_2$; furthermore, equality holds if and only if there exists a re-parameterization of $\mathcal{C}$ which sends p to $(0,0,1)$ and φ to*

$$\varphi' = \begin{bmatrix} T_1 & Q_1 \\ T_2 & Q_2 \\ 0 & Q_3 \end{bmatrix},$$

with the degree sequence for φ' equal to (d_1, d_2), $\gcd(T_1, Q_1) = 1$, $\gcd(T_2, Q_2) = 1$, and $\operatorname{ht} I_2(\varphi') = 2$.

Proof. Let $p\varphi$ be the row vector $[a_1, a_2]$. The General Lemma tells us that

$$m_p = \deg \gcd(a_1, a_2) \le d_2.$$

Henceforth, we assume that $m_p = d_2$. If $d_1 < d_2$, then a_1 is automatically zero. If $d_1 = d_2$, then some non-trivial linear combination of a_1, a_2 is zero. In any event, there are invertible matrices U and V so that $U\varphi V$ has the form

$$\varphi' = \begin{bmatrix} T_1 & Q_1 \\ T_2 & Q_2 \\ 0 & Q_3 \end{bmatrix}.$$

The maximal minors of φ' generate an ideal of height two; hence, T_1, Q_1, Q_3 have no factor in common have no factor in common. One may add a multiple of row 3 to row 1 in order to ensure that the entries of row 1 are relatively prime. The analogous argument works for row 2. $\square$

We must compare Corollary 1.31 to some result to Song Chen Goldman.

Section 2. The Triple Lemma.

Section 3. The Biproj Lemma.

Let φ be a Hilbert-Burch matrix which corresponds to a parameterized plane curve $\mathcal{C}$ of degree d. Points of $\mathbb{P}^2$ give rise to generalized row ideals of the matrix φ. Thus, features of the generalized row ideals reflect properties of the corresponding points. For example, a generalized row ideal of φ encodes information that can be used to determine if the corresponding point p is on $\mathcal{C}$ and, if so, what type of singularity occurs at p.

In this section, we focus on the situation where the degree of $\mathcal{C}$ is even (so $d = 2c$) and we describe the singular points on $\mathcal{C}$ of multiplicity c. We know from Corollary 1.31 that such a point exists if and only if all entries of φ have the same degree c and the corresponding generalized row has a generalized zero. This leads us to consider column operations on φ, which we identify with points in $\mathbb{P}^1$. Thus, inside $\mathbb{P}^2 \times \mathbb{P}^1$ we consider the closed subset consisting of pairs

$$\text{(row operation, column operation)}$$

that lead to a generalized zero of φ. Projection onto the first factor gives the singular points of multiplicity c. On the other hand, projection onto the second factor yields a finite set of points in $\mathbb{P}^1$ that is easier to study yet reflects properties of the set of singular points of multiplicity c on the plane curve $\mathcal{C}$.

Fix a Hilbert-Burch matrix in which every entry is a homogeneous form of degree c. To find a singularity of multiplicity c on $\mathcal{C}$ we need to describe a generalized zero of φ. In other words, we look for (p, q) in $\mathbb{P}^2 \times \mathbb{P}^1$ such that $p\varphi q^{\mathrm{T}} = 0$. Consider the polynomial $\boldsymbol{T}\varphi\boldsymbol{u}^{\mathrm{T}} \in k[\boldsymbol{T}, \boldsymbol{u}, x, y]$, where $\boldsymbol{T} = [T_1, T_2, T_2]$ and $\boldsymbol{u} = [u_1, u_2]$ are matrices of indeterminates. We extract the variables x and y from the critical polynomial $\boldsymbol{T}\varphi\boldsymbol{u}^{\mathrm{T}}$ by the following method. Let $\rho^{(c)} = [y^c, xy^{c-1}, \dots, x^c]$ be the row vector of monomials of degree c in $k[x, y]$. Define C and A to be the matrices with

$$\boldsymbol{T}\varphi = \rho^{(c)}C \quad \text{and} \quad C\boldsymbol{u}^{\mathrm{T}} = A\boldsymbol{T}^{\mathrm{T}},$$

so that the entries of C are linear forms in $k[\boldsymbol{T}]$ and the entries of A are linear forms in $k[\boldsymbol{u}]$. One now has

$$\boldsymbol{T}\varphi\boldsymbol{u}^{\mathrm{T}} = \rho^{(c)}C\boldsymbol{u}^{\mathrm{T}} = \rho^{(c)}A\boldsymbol{T}^{\mathrm{T}}.$$

Thus,

mot (3.1) the set of (p, q) in $\mathbb{P}^2 \times \mathbb{P}^1$ such that $p\varphi q^{\mathrm{T}} = 0$ is the zero set, in $\mathbb{P}^2 \times \mathbb{P}^1$, of the bihomogeneous ideal $I_1(C\boldsymbol{u}^{\mathrm{T}}) = I_1(A\boldsymbol{T}^{\mathrm{T}})$.

In Theorem 3.4, we obtain two isomorphisms of schemes

pict (3.2)

$$\begin{tikzcd} & \operatorname{Biproj}\left(k[\boldsymbol{u},\boldsymbol{T}]/I_1(C\boldsymbol{u}^{\mathrm{T}})\right) \arrow[dl, "\pi_1"', "\cong"] \arrow[dr, "\pi_2", "\cong"'] & \\ \operatorname{Proj}\left(k[\boldsymbol{T}]/I_2(C)\right) & & \operatorname{Proj}\left(k[\boldsymbol{u}]/I_3(A)\right) \end{tikzcd}$$

and we exploit these isomorphisms in Corollary 3.8 to describe the singularities on $\mathcal{C}$ of multiplicity c. The equation $C\boldsymbol{u}^{\mathrm{T}} = A\boldsymbol{T}^{\mathrm{T}}$ provides symmetry. Theorem 3.3 is used twice to produce the isomorphisms of (3.2)

bipro **Theorem 3.3.** *Let* $S = k[x_1, \dots, x_m, y_1, \dots, y_n]$ *be a bi-graded polynomial ring with* $\deg x_i = (1,0)$ *and* $\deg y_i = (0,1)$, *and* R *be the sub-algebra* $k[x_1, \dots, x_m]$ *of* S. *Let* J *be an* S*-ideal generated by bi-homogeneous forms which are linear in the* y*'s. Write* $J = I_1(\phi\boldsymbol{y})$ *where* $\boldsymbol{y} = [y_1, \dots, y_n]^{\mathrm{T}}$ *and* ϕ *is a matrix with entries in* R. *The entries in each row of* ϕ *are homogeneous of the same degree. Consider the natural projection map* $\pi\colon \operatorname{Biproj}(S/J) \to \operatorname{Proj}(R)$. *If the ideal* $I_1(\phi)$ *is zero-dimensional in* R, *then* π *is an isomorphism onto its image and this image is defined scheme-theoretically by the* R*-ideal* $I_n(\phi)$.

Notice that $\operatorname{im}\pi = \operatorname{Proj}\left(R/\left(J : (I_1(\boldsymbol{x})I_1(\boldsymbol{y}))^\infty\right) \cap R\right) \subseteq \operatorname{Proj}(R)$. The theorem means that π gives a bijection $\operatorname{Biproj}(S/J) \to \operatorname{Proj}(R/I_n(\phi))$ which induces isomorphisms at the level of local rings.

Proof. We may dehomogenize with respect to any of the x_i. This has the effect of dehomogenizing the matrix ϕ and hence dehomgenizing the ideals of minors of ϕ. In particular, we have that $I_{n-1}(\phi)$ is the unit ideal. Localizing at any maximal ideal of R we may then assume that R is local and hence a suitable $(n-1)\times(n-1)$ minor of ϕ is invertible. This reduces us to the case where $n = 1$.

Now $S = R[y]$ and $J = \mathfrak{a}yS$, where $\mathfrak{a}$ is the R-ideal $I_1(\phi)$. Notice that $J : y^\infty = \mathfrak{a}yS : y^\infty = \mathfrak{a}S : y^\infty = \mathfrak{a}S$. Therefore, $(J : y^\infty) \cap R = \mathfrak{a} = I_1(\phi)$, which proves the second assertion. To show the first claim notice that the map $R/\mathfrak{a} \to S/\mathfrak{a}yS$ induces an isomorphism $\operatorname{Proj}(S/\mathfrak{a}yS) \to \operatorname{Spec}(R/\mathfrak{a})$. $\square$

XXX **Theorem 3.4.** *Adopt Data 1.5 with* $n = 3$, $d = 2c$, *and* k *an algebraically closed field. Let* φ *be a homogeneous Hilbert-Burch matrix for the row vector* $[g_1, g_2, g_3]$. *Assume that all the entries of* φ *have degree* c. *Let* $\boldsymbol{T}$ *and* $\boldsymbol{u}$ *be the row vectors* $\boldsymbol{T} = [T_1, T_2, T_3]$ *and* $\boldsymbol{u} = [u_1, u_2]$ *of indeterminates. Define the matrices* C *and* A *by*

$$\boldsymbol{T}\varphi = \rho^{(c)}C \quad \textit{and} \quad C\boldsymbol{u}^{\mathrm{T}} = A\boldsymbol{T}^{\mathrm{T}},$$

so that the entries of C are linear forms in $k[\boldsymbol{T}]$ and the entries entries of A are linear forms in $k[\boldsymbol{u}]$. The following statements hold.

(1) *The schemes* $\operatorname{Proj}(\frac{k[\boldsymbol{T}]}{I_2(C)})$ *and* $\operatorname{Proj}(\frac{k[\boldsymbol{u}]}{I_3(A)})$ *are isomorphic.*
(2) *As a subset of* $\mathbb{P}^2$, $\operatorname{Proj}(\frac{k[\boldsymbol{T}]}{I_2(C)})$ *is equal to* $\{p \in \mathcal{C} \mid m_p = c\}$.

Proof. We apply Theorem 3.3 twice. Each time $S = k[\boldsymbol{T}, \boldsymbol{u}]$ and $J = I_1(C\boldsymbol{u}^{\mathrm{T}}) = I_1(A\boldsymbol{T}^{\mathrm{T}})$. In the first application $R = k[\boldsymbol{T}]$. We verify that $I_1(C)$ is zero dimensional ideal of $k[\boldsymbol{T}]$; otherwise, since the entries of C are linear, $I_1(C)$ is contained in an ideal generated by two linear forms. The equation defining C shows that, after row operations, φ has a row of zeros yielding $\operatorname{ht}(I) = 1$. Theorem 3.3 now implies that the schemes $\operatorname{Proj}(k[\boldsymbol{T}]/I_2(C))$ and $\operatorname{Biproj}(S/J)$ are isomorphic.

In the second application $R = k[\boldsymbol{u}]$. We now verify that $I_2(A)$ is a zero dimensional ideal of $k[\boldsymbol{u}]$. Otherwise, $I_2(A)$ is contained in the ideal generated by a homogeneous prime element of $k[u_1, u_2]$. This prime element is a linear form since k is algebraically closed. To make this linear form become u_1 insert an invertible matrix and its inverse between φ and $\boldsymbol{u}^{\mathrm{T}}$ in the critical equation $\boldsymbol{T}\varphi\boldsymbol{u}^{\mathrm{T}} = \rho^{(c)}A\boldsymbol{T}^{\mathrm{T}}$. Reduce modulo u_1 to obtain $\boldsymbol{T}\varphi_2 u_2 = \rho^{(c)}\bar{A}\boldsymbol{T}^{\mathrm{T}}$, where φ_2 is the second column of φ and $\bar{A}$ is A modulo u_1. We have $\bar{A} = u_2 B$ where B is a matrix of scalars. Also, $I_2(\bar{A}) = 0$, so $\operatorname{rank} B \leq 1$. Cancel u_2 to obtain $\boldsymbol{T}\varphi_2^{\mathrm{T}} = \rho^{(c)}B\boldsymbol{T}^{\mathrm{T}}$. Compare the coefficients of T_i to see $\varphi_{i,2} = \rho^{(c)}B_i$, where B_i is the i^{th} column of B. The columns B_1, B_2, B_3 are scalar multiples of each other; and therefore we have obtained the contradiction that $\operatorname{ht} I \leq 1$. Theorem 3.3 now implies that the schemes $\operatorname{Biproj}(S/J)$ and $\operatorname{Proj}(k[\boldsymbol{u}]/I_3(A))$ are isomorphic. Transitivity of isomorphisms yields assertion (1).

Return to the first setting. Theorem 3.3 also yields that the image of the map

t2

$$\mathbb{P}^2 \times \mathbb{P}^1 \supseteq \operatorname{Biproj}(S/J) \xrightarrow{\pi} \operatorname{Proj}(k[\boldsymbol{T}])$$

is $\operatorname{Proj}(k[\boldsymbol{T}]/I_2(C)) \subseteq \operatorname{Proj}(k[\boldsymbol{T}])$. On the other hand, as a set

$$\begin{aligned} \operatorname{im}\pi &= \{p \in \mathbb{P}^2 \mid \exists q \in \mathbb{P}^1 \text{ with } (p,q) \in V(J)\} \\ &= \{p \in \mathbb{P}^2 \mid \exists q \in \mathbb{P}^1 \text{ with } p\varphi q^{\mathrm{T}} = 0\} \qquad (3.5) \\ &= \{p \in \mathcal{C} \mid m_p = c\}. \qquad (3.6) \end{aligned}$$

The equality (3.5) is explained in (3.1) and the equality (3.6) is established in Corollary 1.31. □

blah *Remark 3.7.* The ideals $I_2(C)$ and $I_3(A)$ have dimension at most one because the previous Theorem shows that $\operatorname{Proj}(k[\boldsymbol{T}]/I_2(C))$ is either empty or is a finite set.

TBA **Corollary 3.8.** *Adopt the notation and hypotheses of Theorem 3.4.*

(1) *The ideal* $I_2(C)$ *is zero-dimensional if and only if* $I_3(A)$ *is zero-dimensional if and only if* $\gcd I_3(A)$ *is a unit; otherwise*

$$e(k[\boldsymbol{T}]/I_2(C)) = e(k[\boldsymbol{u}]/I_3(A)) = \deg\gcd I_3(A) = 6 - \mu(I_2(C)).$$

(2) *The non-associate linear forms of* $\gcd I_3(A)$ *correspond to the distinct singular points on* C *of multiplicity* c*.*

(3) *Write* $\gcd(I_3(A)) = \prod \ell_i^{e_i}$*, where the* ℓ_i *are non-associate linear forms and* $e_i \geq 1$*. Then* $e_i - 1$ *is the number of singular points of multiplicity* c *infinitely near to the point on* $\mathcal{C}$ *corresponding to* ℓ_i*.*

(4) *The* $\deg\gcd I_3(A)$ *is the number of distinct singular points of multiplicity* c *that are either on* $\mathcal{C}$ *or infinitely near to* $\mathcal{C}$*.*

Proof. The schemes $\operatorname{Proj}(k[\boldsymbol{T}]/I_2(C))$ and $\operatorname{Proj}(k[\boldsymbol{u}]/I_3(A))$ are isomorphic and are either empty or zero-dimensional by Theorem 3.4 and Remark 3.7. It follows that $k[\boldsymbol{T}]/I_2(C)$ and $k[\boldsymbol{u}]/I_3(A)$ are both zero dimensional or one-dimensional with the same multiplicity. If $\operatorname{ht} I_3(A) = 2$, then $\gcd(I_3 A)$ is a unit. Otherwise, $\operatorname{ht} I_3(A) = 1$ and we can write $I_3 A = (\gcd I_3 A) \cap \mathfrak{q}$ where $\mathfrak{q}$ is a zero-dimensional ideal and

$$e(k[\boldsymbol{u}]/I_3(A)) = e(k[\boldsymbol{u}]/(\gcd I_3 A)) = \deg\gcd I_3 A.$$

SECTION 4. THE GENERIC HILBERT-BURCH MATRIX, BASE POINT FREE LOCUS, AND BIRATIONAL LOCUS.

A resolution of the parameterization of a generic rational plane curve of even degree.

1-6-10 **Proposition 4.1.** *Pick up the notation from the paper Parameter Space. Let $\boldsymbol{g} = (g_1, g_2, g_3)$ be a 3-tuple of homogeneous forms of degree $d = 2c$. Assume that $I_{\boldsymbol{g}}$ has height 2. The following statements hold.*

(1) *There exists a $3c\times 3c$ matrix W such that the degree sequence for the Hilbert-Burch matrix of $\boldsymbol{g}$ is (c,c) if and only if $\det W$ is not zero. Furthermore, each non-zero entry of W is a coefficient from one of the g_i.*

(2) *If $\det W \neq 0$, then there exists a $(3c+1)\times(3c+3)$ matrix A, a 3×3 matrix D_2, and a 3×1 matrix D_3 such that*

(a)

$$0 \to R(-3c) \xrightarrow{D_3} R(-3c)^3 \xrightarrow{D_2} R(-2c)^3 \xrightarrow{G(\boldsymbol{g})} R \to R/I \to 0$$

is a (non-minimal) resolution of R/I,

(b) *each entry of D_3 is a coefficient of a g_i,*

(c) *each non-zero entry of A is a coefficient from one of the g_i,*

(d) *each coefficient of each entry of D_2 is a maximal minor of the matrix A.*

Proof. For each j, with $1 \le j \le 3$, let $g_j = \sum_{i=0}^{d} c_{i,j}x^iy^{d-i}$. For each integer i, let $\rho^{(i)}$ be the $1\times(i+1)$ matrix $[y^i, xy^{i-1}, \dots, x^i]$, $N^{(i)}$ be the $3\times 3(i+1)$ matrix

$$N^{(i)} = \begin{bmatrix} \rho^{(i)} & 0 & 0 \\ 0 & \rho^{(i)} & 0 \\ 0 & 0 & \rho^{(i)} \end{bmatrix},$$

and $A^{(i)}(\boldsymbol{g})$ be the $(d+i+1)\times 3(r+1)$ matrix

$$A^{(i)}(\boldsymbol{g}) = \begin{bmatrix}
c_{0,1} & 0 & \cdots & 0 & c_{0,2} & 0 & \cdots & 0 & c_{0,3} & 0 & \cdots & 0 \\
c_{1,1} & c_{0,1} & \cdots & 0 & c_{1,2} & c_{0,2} & \cdots & 0 & c_{1,3} & c_{0,3} & \cdots & 0 \\
c_{2,1} & c_{1,1} & \cdots & 0 & c_{2,2} & c_{1,2} & \cdots & 0 & c_{2,3} & c_{1,3} & \cdots & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\
c_{d-1,1} & c_{d-2,1} & \cdots & \vdots & c_{d-1,2} & c_{d-2,2} & \cdots & \vdots & c_{d-1,3} & c_{d-2,3} & \cdots & \vdots \\
c_{d,1} & c_{d-1,1} & \cdots & \vdots & c_{d,2} & c_{d-1,2} & \cdots & \vdots & c_{d,3} & c_{d-1,3} & \cdots & \vdots \\
0 & c_{d,1} & \cdots & \vdots & 0 & c_{d,2} & \cdots & \vdots & 0 & c_{d,3} & \cdots & \vdots \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\
0 & 0 & \cdots & c_{d,1} & 0 & 0 & \cdots & c_{d,2} & 0 & 0 & \cdots & c_{d,3}
\end{bmatrix}.$$

Notice that each polynomial g_j contributes exactly $i+1$ columns to $A^{(i)}(\boldsymbol{g})$. Matrix multiplication yields that

key
$$\rho^{(d+r)}A^{(r)}(\boldsymbol{g}) = G(\boldsymbol{g})N^{(r)}, \tag{4.2}$$

for each r. Let $\boldsymbol{q}$ be a 3×1 matrix of forms from R of degree r. Then $\boldsymbol{q} = N^{(r)}\boldsymbol{b}$, where $\boldsymbol{b}$ is a $(3r+3)\times 1$ matrix of scalars. So

$$G(\boldsymbol{g})\boldsymbol{q} = G(\boldsymbol{g})N^{(r)}\boldsymbol{b} = \rho^{(d+r)}A^{(r)}(\boldsymbol{g})\boldsymbol{b}$$

and

ky
$$G(\boldsymbol{g})\boldsymbol{q} = 0 \iff A^{(r)}(\boldsymbol{g})\boldsymbol{b} = 0. \tag{4.3}$$

Now we prove (1). Let W be the $3c\times 3c$ matrix $A^{(c-1)}(\boldsymbol{g})$. Apply (4.3), with $r = c-1$, to see that

$$\begin{array}{ll} & \text{there exists a non-zero } 3\times 1 \text{ matrix } \boldsymbol{q} \text{ of forms of degree } c-1 \text{ with } G(\boldsymbol{g})\boldsymbol{q} = 0 \\ \iff & \text{there exists a non-zero } 3c\times 1 \text{ matrix } \boldsymbol{b} \text{ of constants with } W\boldsymbol{b} = 0 \\ \iff & \det W = 0. \end{array}$$

To prove (2), we take A to be the $(3c+1)\times(3c+3)$ matrix $A^{(c)}(\boldsymbol{g})$. Let $\boldsymbol{q}$ be a non-zero homogeneous minimal syzygy of $G(\boldsymbol{g})$. We know from assertion (1) that every entry of $\boldsymbol{q}$ is a homogeneous form of degree c. Write $\boldsymbol{q} = N^{(c)}\boldsymbol{b}$ for some matrix of scalars $\boldsymbol{b}$. We saw in (4.3) that $A\boldsymbol{b} = 0$. One may obtain $3c+3$ "Eagon-Northcott" relations on A, by crossing one column of A at a time and computing the signed maximal minors of the resulting $(3c+1)\times(3c+2)$ matrix. In particular, when one crosses out columns 1, $c+2$, or $2c+3$ of A, one obtains the relations $\boldsymbol{b}^{(1)}$, $\boldsymbol{b}^{(c+2)}$, and $\boldsymbol{b}^{(2c+3)}$, on A, which are given by

$$\begin{bmatrix} 0 \\ A(1,2) \\ -A(1,3) \\ \vdots \\ (-1)^c A(1,3c+2) \end{bmatrix}, \quad \begin{bmatrix} A(1,c+2) \\ -A(2,c+2) \\ \vdots \\ (-1)^c A(c+1,c+2) \\ 0 \\ (-1)^{c+1}A(c+2,c+3) \\ \vdots \\ (-1)^c A(c+2,3c+2) \end{bmatrix}, \text{ and } \begin{bmatrix} A(1,2c+3) \\ -A(2,2c+3) \\ \vdots \\ -A(2c+2,2c+3) \\ 0 \\ A(2c+3,2c+4) \\ \vdots \\ (-1)^c A(2c+3,3c+2) \end{bmatrix},$$

respectively. The corresponding relations on $G(\boldsymbol{g})$ are

.'quad
$$N^{(c)}\boldsymbol{b}^{(1)}, \quad N^{(c)}\boldsymbol{b}^{(c+2)}, \quad \text{and} \quad N^{(c)}\boldsymbol{b}^{(2c+3)}. \tag{4.4}$$

Let D_2 be the matrix $[\, N^{(c)}\boldsymbol{b}^{(1)} \quad N^{(c)}\boldsymbol{b}^{(c+2)} \quad N^{(c)}\boldsymbol{b}^{(2c+3)} \,]$. We next show that

et

$$R(-3c)^3 \xrightarrow{D_2} R(-2c)^3 \xrightarrow{G(\boldsymbol{g})} R \to R/I_{\boldsymbol{g}} \to 0 \tag{4.5}$$

is exact. Assertion (1) guarantees that the minimal resolution of $R/I_{\boldsymbol{g}}$ looks like

$$0 \to R(-3c)^2 \xrightarrow{\varphi} R(-2c)^3 \xrightarrow{G(\boldsymbol{g})} R \to R/I_{\boldsymbol{g}} \to 0.$$

Compare the vector space, V_1, generated by the columns of D_2, and the vector space, V_2, generated by the columns of φ. It is clear that $V_1 \subseteq V_2$ and that $\dim V_2 = 2$. We show that (4.5) is exact by showing that $\dim V_1$ is at least two. Look at D_2 after x has been set equal to zero:

$$D_2 \cong \begin{bmatrix} 0 & A(1,c+2)y^d & A(1,2c+3)y^d \\ \pm A(1,c+2)y^d & 0 & \pm A(c+2,2c+3)y^d \\ \pm A(1,2c+3)y^d & \pm A(c+2,2c+3)y^d & 0 \end{bmatrix} \mod (x).$$

Expand the minors $A(c+2,2c+3)$, $A(1,2c+3)$, and $A(1,c+2)$ of A across the first row to see that

$$A(c{+}2,2c{+}3) = c_{0,1}\det W, \quad A(1,2c{+}3) = c_{0,2}\det W, \text{ and } A(1,c+2) = c_{0,3}\det W.$$

Thus,

$$D_2 \cong \det W y^d \begin{bmatrix} 0 & c_{0,3} & c_{0,2} \\ \pm c_{0,3} & 0 & \pm c_{0,1} \\ \pm c_{0,2} & \pm c_{0,1} & 0 \end{bmatrix} \mod (x).$$

The determinant of W is not zero by hypothesis. The ideal $I_{\boldsymbol{g}}$ has height 2; so, at least one of the coefficients $c_{0,1}$, $c_{0,2}$, and $c_{0,3}$ is non-zero. It follows that the column space V_1 of D_2 has dimension at least two and (4.5) is exact.

We next identify the kernel of D_2. We know, a priori, that the kernel of D_2 is generated by a single non-trivial column vector of constants. Once again we consider the Eagon-Northcott complex associated to the map $A\colon R^{3c+3} \to R^{3c+1}$:

$$0 \to R^{3c+1} \otimes \bigwedge^{3c+3} R^{3c+3} \to \bigwedge^{3c+2} R^{3c+3} \to R^{3c+3} \xrightarrow{A} R^{3c+1}.$$

Each row of $A(\boldsymbol{g})$ gives rise to a relation on the $3c+3$ Eagon-Northcott relations on A. In particular, the top row of A gives rise to the relation

$$c_{0,1}\boldsymbol{b}^{(1)} - c_{0,2}\boldsymbol{b}^{(c+2)} + c_{0,3}\boldsymbol{b}^{(2c+3)} = 0.$$

It follows that

$$D_3 = \begin{bmatrix} c_{0,1} \\ c_{0,2} \\ c_{0,3} \end{bmatrix}$$

is a non-trivial relation on D_2. The proof is complete. $\square$

(1) Can one identify the "no base point locus" using polynomial equations in the coefficients of the parameterizing equations for general degree?
(2) Can one identify the "birational locus" using polynomial equations in the coefficients of the parameterizing equations for general degree?
(3) This generic hilbert burch matrix says something about flatness. What does it say?
(4) What can be done when the degree is odd?
(5) In the above argument $\det W \neq 0$ already ensures no base points!

Section 5. Open and closed loci.

Section 6. The branches of a parameterized curve are paramterized.

Section 7. The Jacobian matrix and the ramification locus.

Remark 1.30 provides a method of parameterizing the branches of a parameterized curve. Theorem 7.2 shows that the Jacobian matrix associated to the parameterization identifies the non-smooth branches of the curve as well as the multiplicity along each branch. The starting point for this line of reasoning is the result that if D is an algebra which is essentially of finite type over the ring C, then the ramification locus of D over C is equal to the support of the module of Kähler differentials $\Omega_{D/C}$. (See, for example, [6, Cor. 6.10].) In our ultimate application of this result, we consider the module of differentials Ω for the ring extension

re

$$\frac{\hat{R}}{J_i} \to \hat{S}_{\mathfrak{M}_i} \tag{7.1}$$

from the proof of Lemma 1.7. We have two presentations of the $\hat{S}_{\mathfrak{M}_i}$-module Ω. One presentation comes from the Jacobian matrix of the parameterization of the curve $\mathcal{C}$. The other presentation comes from the geometry which gives rise to the ring extension (7.1). The Fitting ideal $\operatorname{Fitt}_0(\Omega)$ may be computed using either presentation.

In addition to [6] one may consult [2, Chapt. 16] or [10, Sect. 26] for elementary facts and notation pertaining to Kähler differentials.

,T4.1 **Theorem 7.2.** *Adopt the Data of 1.5 with k an algebraically closed field of characteristic zero. Consider the inclusion map $k[I_d] \subseteq B$ of homogeneous coordinate rings which is induced by the morphism $\Psi\colon \mathbb{P}^1 \to \mathcal{C}$. The* gcd *of the Fitting ideal of $\Omega_{B/k[I_d]}$ is a polynomial in B. Let*

$$\operatorname{gcd} \operatorname{Fitt}_0(\Omega_{B/k[I_d]}) = \prod_{i=1}^{s} \ell_i^{f_i},$$

where $(\ell_1), \dots, (\ell_s)$ are distinct linear ideals of B. If (ℓ) is an arbitrary linear ideal of B and $\mathcal{C}(\ell)$ is the branch of $\mathcal{C}$ which corresponds to ℓ, in the sense of Remark 1.30, then the multiplicity of $\mathcal{C}$ along the branch $\mathcal{C}(\ell)$ is

$$\begin{cases} f_i + 1 & \text{if } (\ell) = (\ell_i) \text{ for some } i \\ 1 & \text{otherwise.} \end{cases}$$

Furthermore, the Fitting ideal $\operatorname{Fitt}_0(\Omega_{B/k[I_d]})$ *is equal to the ideal* $I_2(N)$ *of* B*, where* N *is the* $2 \times n$ *Jacobian matrix*

$$N = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \cdots & \frac{\partial g_n}{\partial x} \\ \frac{\partial g_1}{\partial y} & \cdots & \frac{\partial g_n}{\partial y} \end{bmatrix}.$$

Before proving the Theorem 7.2, we describe various special cases. Corollary 7.3 follows immediately from Theorem 7.2 without any further proof. Also, Theorem 0.2 requires only a small amount of additional proof.

cl **Corollary 7.3.** *Retain the notation and hypotheses of Theorem 7.2.*

(1) *All of the branches through all of the points of* $\mathcal{C}$ *are smooth if and only if* $\operatorname{ht} I_2(N) = 2$.
(2) *The multiplicity along each branch of* $\mathcal{C}$ *is at most two if and only if the* gcd *of* $I_2(N)$ *has distinct linear factors.*

Proof of Theorem 0.2. We are given the singular points $p_1, \dots, p_z$ on $\mathcal{C}$ and the factorizations $\gcd I_1(p_j\varphi) = \prod \ell_{u,j}^{e_{u,j}}$. Lemma 1.7 tells us that the multiplicity at p_j along the branch $\mathcal{C}(\ell_{u,j})$ of $\mathcal{C}$ is $e_{u,j}$. Assertion (1) is now an immediate consequence Theorem 7.2. We prove (2). Theorem 0.1 shows that for each j, with $1 \le j \le z$, the polynomial $\gcd I_1(p_j\varphi)$ has degree m_j and s_j distinct linear factors; so, one has

$$\deg \prod_{u=1}^{s_j} \ell_{u,j}^{e_{u,j}-1} = \deg \prod_{u=1}^{s_j} \ell_{u,j}^{e_{u,j}} - s_j = m_j - s_j;$$

hence, (1) gives

$$\deg \gcd I_2(N) = \deg \prod_{j=1}^{z} \prod_{u=1}^{s_j} \ell_{u,j}^{e_{u,j}-1} = \sum_{j=1}^{z} (m_j - s_j). \quad \square$$

Proof of Theorem 7.2. The relative cotangent complex

$$\Omega_{k[I_d]/k} \otimes_k B \to \Omega_{B/k} \to \Omega_{B/k[I_d]} \to 0$$

gives rise to the presentation

,.prez (7.4) $$B^n \xrightarrow{N} B^2 \to \Omega_{B/k[I_d]} \to 0$$

of $\Omega_{B/k[I_d]}$ as a B-module. It follows immediately that the Fitting ideal of the B-module $\Omega_{B/k[I_d]}$ is

$$\operatorname{Fitt}_0(\Omega_{B/k[I_d]}) = I_2(N)B.$$

Fix a point $q \in \mathbb{P}^1$ and a non-zero linear form $\ell \in B$ with $\ell(q) = 0$. Let p be the point $(g_1(q), \dots, g_n(q))$ on $\mathcal{C}$ in $\mathbb{P}^{n-1}$ and e be the multiplicity of $\mathcal{C}$ along the branch $\mathcal{C}(\ell)$. Define f to be the exponent with

,exp (7.5)
$$\gcd I_2(N) = \ell^f \cdot \theta,$$

where θ is a polynomial in B which is relatively prime to ℓ. We prove $e = f + 1$. Let $\mathfrak{p}$ be the ideal

$$I_2 \begin{pmatrix} g_1(q) & \dots & g_n(q) \\ g_1 & \dots & g_n \end{pmatrix}$$

of $k[I_d]$.

The formation of Ω is preserved under base change. That is, if C' and D are C-algebras, then

$$C' \otimes_C \Omega_{D/C} = \Omega_{(C' \otimes_C D)/C'}.$$

In particular, any presentation of $\Omega_{D/C}$ as an D-module:

$$D^a \xrightarrow{\sigma} D^b \xrightarrow{\tau} \Omega_{D/C} \to 0$$

gives rise to a presentation of $\Omega_{(C' \otimes_C D)/C'}$ as an $C' \otimes_C D$-module:

$$(C' \otimes_C D)^a \xrightarrow{\sigma} (C' \otimes_C D)^b \xrightarrow{\tau} C' \otimes_C \Omega_{D/C} = \Omega_{(C' \otimes_C D)/C'} \to 0.$$

For example, if $C' = U^{-1}C$ for some multiplicatively closed subset U of C, then we write $U^{-1}D$ in place of $U^{-1}C \otimes_C D$; so we have the presentation

$$(U^{-1}D)^a \xrightarrow{\sigma} (U^{-1}D)^b \xrightarrow{\tau} \Omega_{U^{-1}D/U^{-1}C} \to 0.$$

In our situation, we localize at $U = k[I_d] \setminus \mathfrak{p}$. We write $k[I_d]_{\mathfrak{p}}$ in place of $U^{-1}k[I_d]$ and $B_{\mathfrak{p}}$ in place of $U^{-1}(B)$. Apply the base change $k[I_d]_{\mathfrak{p}} \otimes_{k[I_d]} \underline{\quad}$ to (7.4) to obtain the following presentation by free $B_{\mathfrak{p}}$-modules

,.nonum (7.6)
$$B_{\mathfrak{p}}^n \xrightarrow{N} B_{\mathfrak{p}}^2 \to \Omega_{B_{\mathfrak{p}}/k[I_d]_{\mathfrak{p}}} \to 0.$$

In the statement of Lemma 1.7 we have called $k[I_d]_{\mathfrak{p}} = R \subseteq S = B_{\mathfrak{p}}$. The ring R is local with maximal ideal $\mathfrak{m}_R = \mathfrak{p}R_{\mathfrak{p}}$. In this new language, (7.6) becomes the exact sequence of S-modules:

,.non1 (7.7)
$$S^n \xrightarrow{N} S^2 \to \Omega_{S/R} \to 0.$$

Complete both $R \subseteq S$ in the $\mathfrak{m}_R$-adic topology to obtain the rings $\hat{R} \subseteq \hat{S}$. One of the maximal ideals of $\hat{S}$ is $(\ell)\hat{S}$, which we denote by $\mathfrak{M}$. Let J be the kernel of $\hat{R} \to \hat{S}_{\mathfrak{M}}$. It is shown at (1.19) that the multiplicity $e(\hat{R}/J)$ is equal to e, which is the multiplicity of $\mathcal{C}$ along the branch $\mathcal{C}(\ell)$.

Apply the base change $\hat{R}\otimes_R \_\_$ to (7.7) to obtain the exact sequence of $\hat{R}\otimes_R S = \hat{S}$ modules:

$$\hat{S}^n \xrightarrow{N} \hat{S}^2 \to \Omega_{\hat{S}/\hat{R}} \to 0.$$

Localize at the multiplicatively closed set $\hat{S} \setminus \mathfrak{M}$ of $\hat{S}$ to obtain an exact sequence of $\hat{S}_{\mathfrak{M}}$-modules:

$$\hat{S}^n_{\mathfrak{M}} \xrightarrow{N} \hat{S}^2_{\mathfrak{M}} \to \Omega_{\hat{S}_{\mathfrak{M}}/\hat{R}} \to 0.$$

Apply the base change $\hat{R}/J \otimes_{\hat{R}} \_\_$. Keep in mind that $\hat{R}/J \otimes_{\hat{R}} \hat{S}_{\mathfrak{M}} = \hat{S}_{\mathfrak{M}}$. Obtain an exact sequence of $\hat{S}_{\mathfrak{M}}$-modules

$$\hat{S}^n_{\mathfrak{M}} \xrightarrow{N} \hat{S}^2_{\mathfrak{M}} \to \Omega_{\hat{S}_{\mathfrak{M}}/\frac{\hat{R}}{J}} \to 0.$$

The Fitting ideal of $\Omega_{\hat{S}_{\mathfrak{M}}/\frac{\hat{R}}{J}}$ is

,Fitt1 (7.8) $$\mathrm{Fitt}_0(\Omega_{\hat{S}_{\mathfrak{M}}/\frac{\hat{R}}{J}}) = I_2(N)\hat{S}_{\mathfrak{M}} = (\ell^f)\hat{S}_{\mathfrak{M}} = \mathfrak{M}^f\hat{S}_{\mathfrak{M}}.$$

Now we calculate $\Omega_{\hat{S}_{\mathfrak{M}}/\frac{\hat{R}}{J}}$ in a completely different manner. Recall the Veronese ring $k[B_d]$ and the ring $T = k[B_d]_{\mathfrak{p}}$ from Lemma 1.7. The completion of T in the $\mathfrak{m}_R$-adic topology is denoted $\hat{T}$. We have

$$\hat{R}/J \subseteq \hat{T}_{\mathfrak{M}\cap\hat{T}} \subseteq \hat{S}_{\mathfrak{M}},$$

with $\hat{T}_{\mathfrak{M}\cap\hat{T}}$ equal to the normalization of $\hat{R}/J$. The rings $\hat{T}_{\mathfrak{M}\cap\hat{T}}$ and $\hat{R}/J$ share the same residue class field, which, in the language of the proof of Lemma 1.7, was called $k(g_{i_0})$. Furthermore, $k(g_{i_0}) \subseteq \hat{R}/J$. The rings $\hat{T}_{\mathfrak{M}\cap\hat{T}}$ and $\hat{S}_{\mathfrak{M}}$ are complete DVRs with the same uniformizing parameter $t = \frac{\ell}{m}$, where m is any linear form in B for which ℓ, m is a basis for the vector space B_1. The ring $\hat{T}_{\mathfrak{M}\cap\hat{T}}$ is equal to $k(m^d)[[t]]$ and the ring $\hat{S}_{\mathfrak{M}}$ is equal to $k(m)[[t]]$. It was observed above that $e(\hat{R}/J) = e$. Proposition 7.10 shows that $\Omega_{\hat{S}_{\mathfrak{M}}/\frac{\hat{R}}{J}}$ is isomorphic to $\hat{S}_{\mathfrak{M}}/(t^{e-1})$. We conclude that the Fitting ideal of $\Omega_{\hat{S}_{\mathfrak{M}}/\frac{\hat{R}}{J}}$ is also equal to

,Fitt2 (7.9) $$\mathrm{Fitt}_0(\Omega_{\hat{S}_{\mathfrak{M}}/\frac{\hat{R}}{J}}) = (t^{e-1})\hat{S}_{\mathfrak{M}} = \mathfrak{M}^{e-1}\hat{S}_{\mathfrak{M}}.$$

Compare (7.8) and (7.9) to see that $f = e - 1$. $\square$

.P33.16 **Proposition 7.10.** *Let $K \subseteq A \subseteq B \subseteq C$ be local rings. Assume that*

(1) *$B = L[[t]]$ and $C = M[[t]]$ are formal power series rings in one variable where $L \subseteq M$ are fields of characteristic zero with $\dim_L M$ finite,*
(2) *K is a field and the natural maps*

$$K \to A \to A/\mathfrak{m}_A \quad \textit{and} \quad K \to B \to B/(t) = L$$

are isomorphisms,
(3) *B finitely generated as an A-module, and*
(4) *$B \subseteq \operatorname{qf}(A)$.*

Then the C-modules $\Omega_{C/A}$ and $\frac{C}{(t^{e-1})C}$ are isomorphic, where $e = e(A)$ is the multiplicity of the local ring A.

Proof. The relative cotangent sequence gives an exact sequence of C-modules:

$$\Omega_{A/K} \otimes_A C \xrightarrow{\alpha} \Omega_{C/K} \xrightarrow{\beta} \Omega_{C/A} \to 0,$$

where $\alpha(da \otimes c) = (da)c$ and $\beta(dc) = dc$. The ring C is generated as an A-algebra by t together with a finite generating set for M over K; so, $\Omega_{C/A}$ is finitely generated as a C-module. The ring C is local; so the Krull Intersection Theorem guarantees that β sends $\cap(t^n)\Omega_{C/K}$ to zero. Thus, the above exact sequence induces the exact sequence of C-modules

$$\Omega_{A/K} \otimes_A C \xrightarrow{\bar\alpha} \overline{\Omega}_{C/K} \xrightarrow{\bar\beta} \Omega_{C/A} \to 0,$$

where

$$\overline{\Omega}_{C/K} = \frac{\Omega_{C/K}}{\cap(t^n)\Omega_{C/K}}$$

and $\bar\alpha$ and $\bar\beta$ are induced by α and β. If $\omega \in \Omega_{C/K}$, then we write $\bar\omega$ for the image of ω in $\overline{\Omega}_{C/K}$. The field extension $K \subseteq M$ is separable and algebraic so the universal derivation $d = d_{C/K} \colon C \to \Omega_{C/K}$ sends M to zero. Therefore, if $f \in C$, then the elements df and $f'dt$ of $\Omega_{C/K}$ represent the same class in $\overline{\Omega}_{C/K}$. It follows that $\overline{\Omega}_{C/K}$ is generated as a C-module by $\overline{dt}$. To complete the proof we show that

(1) $\overline{\Omega}_{C/K}$ is a free C-module and
(2) the image of $\bar\alpha$ is equal to $(t^{e-1})C\overline{dt}$.

We prove (2) first. The one-dimensional domains $A \subseteq B$ are local and B is the normalization of A; so, Observation 1.21 guarantees the existence of an element z in $\mathfrak{m}_A$ such that $z = t^e +$ higher order terms and $\mathfrak{m}_A B = zB$. It follows that

$\mathfrak{m}_A C = zC$. The image of $\bar{\alpha}$ is the C-submodule of $\overline{\Omega}_{C/K} = C\overline{dt}$ which is generated by $\bar{\alpha}(dz)$ and this is the equal to $(t^{e-1})C\overline{dt}$ since the field K has characteristic zero.

Now we prove (1). Suppose that $\theta \in C$ and that the element θdt of $\Omega_{C/K}$ is in $\bigcap_{i=0}^{\infty}(t^i)\Omega_{C/K}$. We prove that θ is zero in C. Let n be a positive integer. Consider the conormal exact sequence

.con (7.11)
$$(t^n)/(t^{2n}) \to \Omega_{C/K} \otimes_C C/(t^n) \to \Omega_{\frac{C}{(t^n)}/K} \to 0$$

of $C/(t^n)$-modules associated to the K-algebra homomorphism $C \to C/(t^n)$. The sequence (7.11) induces an isomorphism

.osi (7.12)
$$\frac{\Omega_{C/K}}{(t^{n-1})Cdt + (t^n)C\Omega_{C/K}} \xrightarrow{\cong} \Omega_{\frac{C}{(t^n)}/K},$$

which is given by class of $df \mapsto d(\text{class of } f)$, for all $f \in C$. The element θdt of $\Omega_{C/K}$ represents the class of zero in the module on the left side of (7.12); so $\bar{\theta}dt$ is zero in $\Omega_{\frac{C}{(t^n)}/K}$, where $\bar{\theta}$ is the image of θ in $C/(t^n)$. On the other hand, $C/(t^n) = M[t]/(t^n)$ and it is well-known that $1 \mapsto dt$ gives an isomorphism

$$\tfrac{M[t]}{(t^{n-1})} \xrightarrow{\cong} \Omega_{\frac{M[t]}{(t^n)}/K}.$$

Thus, the image $\bar{\theta}$ of θ in $C/(t^{n-1})$ is zero. This process may be repeated for all n. We conclude that θ is zero in C. $\square$

SECTION 8. AN AFFINE PARAMETERIZATION OF THE SPACE OF RATIONAL PLANE CURVES OF DEGREE FOUR.

The configuration of singularities that can appear on a rational plane quartic are completely determined by two classical formulas:

$$g = \binom{d-1}{2} - \sum_q \binom{m_q}{2} \qquad \text{and} \qquad \sum_{q'} m_{q'} \le m_p,$$

The formula on the left was known by Max Noether. It gives the genus g of the irreducible plane curve $\mathcal{C}$ of degree d, where q varies over all visible and "infinitely near" singularities of $\mathcal{C}$, and m_q is the singularity multiplicity at q. The formula on the right holds whenever the curve $\mathcal{C}''$ is obtained by blowing-up the curve $\mathcal{C}'$ at a singular point p. In this formula q' varies over all points on $\mathcal{C}''$ which lie over p. The above formulas permit 9 possible visible singularities on a rational plane curve of degree 4:

Classical name	modern name	multiplicity sequence	(m,δ,s)
Node	A_1	$(2:1,1)$	$(2,1,2)$
Cusp	A_2	$(2:1)$	$(2,1,1)$
Tacnode	A_3	$(2:2:1,1)$	$(2,2,2)$
Ramphoid Cusp	A_4	$(2:2:1)$	$(2,2,1)$
Oscnode	A_5	$(2:2:2:1,1)$	$(2,3,2)$
A_6-cusp	A_6	$(2:2:2:1)$	$(2,3,1)$
Ordinary Triple Point	D_4	$(3:1,1,1)$	$(3,3,3)$
Tacnode Cusp	D_5	$(3:1,1)$	$(3,3,2)$
Multiplicity 3 cusp	E_6	$(3:1)$	$(3,3,1)$

(The terminology "multiplicity sequence" and "(m,δ,s)" are described later in this introduction.) There are thirteen possible ways to configure the above singularities on a rational plane quartic. (The complete list is given below.) This classification has been know for well over one hundred years; see, for example, Basset [1], Hilton [5], Namba [11], or Wall [12]. The singularity "A_6" does not have a consistent classical name and for that reason we have introduced the name "A_6-cusp". Beware that Namba's terminology is not completely consistent with the terminology used above: he uses "double cusp" (respectively, "ramphoid cusp") for what we call a ramphoid cusp (resp. "A_6-cusp").

Consider an ordered triple $\boldsymbol{g} = (g_1, g_2, g_3)$ of homogeneous polynomials of degree four in two variables. The triple $\boldsymbol{g}$ parameterizes a curve $\mathcal{C}_{\boldsymbol{g}}$ in $\mathbb{P}^2$. Let $\mathbb{A}$ be the set of all such triples. In this note we identify various closed subsets of the affine space $\mathbb{A}$ and we describe the singularities of $\mathcal{C}_{\boldsymbol{g}}$ as a function of where $\boldsymbol{g}$ sits in $\mathbb{A}$.

Indeed, we identify an open subset $\mathbb{U}'$ of $\mathbb{A}$ with the property that every rational plane quartic $\mathcal{C}$ is parameterized by some $\boldsymbol{g} \in \mathbb{U}'$. (In this case we write $\mathcal{C}_{\boldsymbol{g}}$ for $\mathcal{C}$.) Furthermore, we identify closed subsets S_3, and $S_4 \subset S_5$, and $C_1 \subseteq C_2 \subseteq C_3$ of $\mathbb{A}$ such that if $\boldsymbol{g} \in \mathbb{U}'$, then

VS

$$\begin{array}{ll} \mathcal{C}_{\boldsymbol{g}} \text{ has 3 visible singularities} & \iff \boldsymbol{g} \in \mathrm{VS}(3), \\ \mathcal{C}_{\boldsymbol{g}} \text{ has 2 visible singularities} & \iff \boldsymbol{g} \in \mathrm{VS}(2), \\ \mathcal{C}_{\boldsymbol{g}} \text{ has 1 visible singularity of multiplicity 2} & \iff \boldsymbol{g} \in \mathrm{VS}_2(1), \\ \mathcal{C}_{\boldsymbol{g}} \text{ has 1 visible singularity of multiplicity 3} & \iff \boldsymbol{g} \in \mathrm{VS}_3(1), \end{array} \tag{8.1}$$

where

$$\begin{aligned} \mathrm{VS}(3) &= \mathbb{U}' \cap (\mathbb{A} \setminus S_3) \cap (\mathbb{A} \setminus S_5) \\ \mathrm{VS}(2) &= \mathbb{U}' \cap (\mathbb{A} \setminus S_3) \cap (\mathbb{A} \setminus S_4) \cap S_5 \\ \mathrm{VS}_2(1) &= \mathbb{U}' \cap (\mathbb{A} \setminus S_3) \cap S_4 \\ \mathrm{VS}_3(1) &= \mathbb{U}' \cap S_3 \cap S_4. \end{aligned}$$

In particular, "$\mathcal{C}_{\boldsymbol{g}}$ has 3 visible singularities" is an open condition and "$\mathcal{C}_{\boldsymbol{g}}$ has 1 visible singularity of multiplicity 3" is a closed condition. Furthermore, a more precise description of the singularities of $\mathcal{C}_{\boldsymbol{g}}$ as a function of where $\boldsymbol{g}$ sits in $\mathbb{A}$ is

A also available. Indeed, if $\boldsymbol{g} \in \mathbb{U}'$, then

$$\begin{array}{ll} \mathcal{C}_{\boldsymbol{g}} \text{ has 3 nodes} & \iff \boldsymbol{g} \in \mathrm{VS}(3) \cap (\mathbb{A} \setminus C_1) \\ \mathcal{C}_{\boldsymbol{g}} \text{ has 2 nodes and 1 cusp} & \iff \boldsymbol{g} \in \mathrm{VS}(3) \cap C_1 \cap (\mathbb{A} \setminus C_2) \\ \mathcal{C}_{\boldsymbol{g}} \text{ has 1 node and 2 cusps} & \iff \boldsymbol{g} \in \mathrm{VS}(3) \cap C_2 \cap (\mathbb{A} \setminus C_3) \\ \mathcal{C}_{\boldsymbol{g}} \text{ has 3 cusps} & \iff \boldsymbol{g} \in \mathrm{VS}(3) \cap C_3 \end{array}$$

$$\begin{array}{ll} \mathcal{C}_{\boldsymbol{g}} \text{ has 1 Tacnode and 1 node} & \iff \boldsymbol{g} \in \mathrm{VS}(2) \cap (\mathbb{A} \setminus C_1) \\ \left\{ \begin{array}{c} \mathcal{C}_{\boldsymbol{g}} \text{ has 1 Tacnode and 1 cusp} \\ \text{or} \\ \mathcal{C}_{\boldsymbol{g}} \text{ has 1 Ramphoid Cusp and 1 node} \end{array} \right\} & \iff \boldsymbol{g} \in \mathrm{VS}(2) \cap C_1 \cap (\mathbb{A} \setminus C_2) \\ \mathcal{C}_{\boldsymbol{g}} \text{ has 1 Ramphoid Cusp and 1 cusp} & \iff \boldsymbol{g} \in \mathrm{VS}(2) \cap C_2 \end{array}$$

$$\begin{array}{ll} \mathcal{C}_{\boldsymbol{g}} \text{ has 1 Oscnode} & \iff \boldsymbol{g} \in \mathrm{VS}_2(1) \cap (\mathbb{A} \setminus C_1) \end{array}$$

$\mathcal{C}_{\boldsymbol{g}}$ has 1 A_6-Cusp	$\iff \boldsymbol{g} \in \mathrm{VS}_2(1) \cap C_1$

$\mathcal{C}_{\boldsymbol{g}}$ has 1 Ordinary Triple Point	$\iff \boldsymbol{g} \in \mathrm{VS}_3(1) \cap (\mathbb{A} \setminus C_1)$
$\mathcal{C}_{\boldsymbol{g}}$ has 1 Tacnode Cusp	$\iff \boldsymbol{g} \in \mathrm{VS}_3(1) \cap C_1 \cap (\mathbb{A} \setminus C_2)$
$\mathcal{C}_{\boldsymbol{g}}$ has 1 Multiplicity 3 cusp	$\iff \boldsymbol{g} \in \mathrm{VS}_3(1) \cap C_2$

Table 8.2

In particular, "$\mathcal{C}_{\boldsymbol{g}}$ has 3 nodes" is an open condition and "$\mathcal{C}_{\boldsymbol{g}}$ has 1 multiplicity 3 cusp" is a closed condition. The proof of the assertions of Table 8.2 are carried out in Section 1; see in particular Lemma 8.22

Consider an ordered triple $\boldsymbol{g} = (g_1, g_2, g_3)$ of homogeneous polynomials of degree four in two variables. The triple $\boldsymbol{g}$ parameterizes a curve $\mathcal{C}_{\boldsymbol{g}}$ in $\mathbb{P}^2$. Let $\mathbb{A}$ be the set of all such triples. In this note we identify various closed subsets S_i and C_i of the affine space $\mathbb{A}$ and we describe the singularities of $\mathcal{C}_{\boldsymbol{g}}$ as a function of where $\boldsymbol{g}$ sits in $\mathbb{A}$. The subsets S_i and C_i are defined in Definition 8.6. The main result is Theorem 8.8.

.'Org **Data 8.3.** Let k be an algebraically closed field, V be a two-dimensional vector space over k, and R be the polynomial ring $R = \operatorname{Sym}_\bullet V$. Let

$$\mathbb{A} = \operatorname{Sym}_4 V \times \operatorname{Sym}_4 V \times \operatorname{Sym}_4 V.$$

Remark. Observe that $\mathbb{A}$ is an affine space over k of dimension 15. Each element of $\mathbb{A}$ has the form $\boldsymbol{g} = (g_1, g_2, g_3)$, where $\boldsymbol{g}$ is an ordered 3-tuple and $g_i \in \operatorname{Sym}_4 V$.

.'D27.13 **Definition 8.4.** Adopt Data 8.3. If $\boldsymbol{g} = (g_1, g_2, g_3) \in \mathbb{A}$, then let $I_{\boldsymbol{g}}$ be the ideal $(g_1, g_2, g_3)R$ of R and $\Psi_{\boldsymbol{g}}$ be the morphism

$$\Psi_{\boldsymbol{g}} \colon \mathbb{P}^1 \setminus X(I_{\boldsymbol{g}}) \to \mathbb{P}^2,$$

which is given by

$$\Psi_{\boldsymbol{g}}(\mathrm{pt}) = (g_1(\mathrm{pt}), g_2(\mathrm{pt}), g_3(\mathrm{pt})),$$

where $X(I_{\boldsymbol{g}})$ is the zero locus in $\mathbb{P}^1$ of $I_{\boldsymbol{g}}$, and $\mathcal{C}_{\boldsymbol{g}}$ be the closure of the image of $\Psi_{\boldsymbol{g}}$.

Note. In the language of Data 8.3, a point of $\mathbb{P}^1$ is a one-dimensional subspace of V^*.

In Definition 8.6 we identify eight subsets $S_1, \dots, S_5$, with $S_4 \subseteq S_5$, and $C_3 \subseteq C_2 \subseteq C_1$ of $\mathbb{A}$. In Lemma 8.7 we demonstrate that each S_i and each C_i is a closed subset of $\mathbb{A}$. We define $\mathbb{U} \subseteq \mathbb{U}'$ to be the open subsets

$$\mathbb{U} = \mathbb{A} \setminus (S_1 \cup S_2 \cup S_3) \subseteq \mathbb{U}' = \mathbb{A} \setminus (S_1 \cup S_2)$$

of $\mathbb{A}$. In practice we are only interested in the open subset $\mathbb{U}'$ of $\mathbb{A}$. Every element $\boldsymbol{g}$ of $\mathbb{A}$ which is not in $\mathbb{U}'$ corresponds to an unsuitable parameterization of the curve $\mathcal{C}_{\boldsymbol{g}}$. (This fact is established in Lemma 8.10.) Some of these unsuitable parameterizations have base points; others are not birational. One can remove base points by factoring out and removing the greatest common factor of the parameterizing forms. Also, if the parameterization is not birational, then one can reparameterize to find a birational parameterization (necessarily of smaller degree). The geometry of the curve $\mathcal{C}_{\boldsymbol{g}}$, as a function of where $\boldsymbol{g}$ sits inside $\mathbb{A}$, is described in Theorem 8.8.

.D28.22 **Definition 8.5.** Adopt Data 8.3. Fix coordinates on $\mathbb{A}$ by picking a basis x, y for V. Fix a point $\boldsymbol{g} = (g_1, g_2, g_3) \in \mathbb{A}$. The corresponding point in $\mathbb{A}^{15}$ is

$$(c_{0,1}, c_{1,1}, c_{2,1}, c_{3,1}, c_{4,1}, c_{0,2}, \dots, c_{4,3}),$$

where

$$g_j = \sum_{i=0}^{4} c_{i,j} x^i y^{4-j},$$

for $c_{i,j} \in k$. Let $G(\boldsymbol{g})$ represent the row vector $[\, g_1 \quad g_2 \quad g_3 \,]$.

(a) Let $M(\boldsymbol{g})$ be the 9×6 matrix with entries from k with

$$\begin{aligned}&[\, y^8 \quad xy^7 \quad x^2y^6 \quad x^3y^5 \quad x^4y^4 \quad x^5y^3 \quad x^6y^2 \quad x^7y \quad x^8 \,]\, M(\boldsymbol{g})\\ &\qquad = [\, g_1^2 \quad g_1g_2 \quad g_1g_3 \quad g_2^2 \quad g_1g_3 \quad g_3^2 \,].\end{aligned}$$

(b) Let $W(\boldsymbol{g})$ be the 6×6 matrix

$$W(\boldsymbol{g}) = \begin{bmatrix} c_{0,1} & 0 & c_{0,2} & 0 & c_{0,3} & 0 \\ c_{1,1} & c_{0,1} & c_{1,2} & c_{0,2} & c_{1,3} & c_{0,3} \\ c_{2,1} & c_{1,1} & c_{2,2} & c_{1,2} & c_{2,3} & c_{1,3} \\ c_{3,1} & c_{2,1} & c_{3,2} & c_{2,2} & c_{3,3} & c_{2,3} \\ c_{4,1} & c_{3,1} & c_{4,2} & c_{3,2} & c_{4,3} & c_{3,3} \\ 0 & c_{4,1} & 0 & c_{4,2} & 0 & c_{4,3} \end{bmatrix},$$

with entries from k.

(c) Let $A(\boldsymbol{g})$ be the 7×9 matrix

$$A(\boldsymbol{g}) = \begin{bmatrix} c_{0,1} & 0 & 0 & c_{0,2} & 0 & 0 & c_{0,3} & 0 & 0 \\ c_{1,1} & c_{0,1} & 0 & c_{1,2} & c_{0,2} & 0 & c_{1,3} & c_{0,3} & 0 \\ c_{2,1} & c_{1,1} & c_{0,1} & c_{2,2} & c_{1,2} & c_{0,2} & c_{2,3} & c_{1,3} & c_{0,3} \\ c_{3,1} & c_{2,1} & c_{1,1} & c_{3,2} & c_{2,2} & c_{1,2} & c_{3,3} & c_{2,3} & c_{1,3} \\ c_{4,1} & c_{3,1} & c_{2,1} & c_{4,2} & c_{3,2} & c_{2,2} & c_{4,3} & c_{3,3} & c_{2,3} \\ 0 & c_{4,1} & c_{3,1} & 0 & c_{4,2} & c_{3,2} & 0 & c_{4,3} & c_{3,3} \\ 0 & 0 & c_{4,1} & 0 & 0 & c_{4,2} & 0 & 0 & c_{4,3} \end{bmatrix},$$

with entries from k.

(d) If $1 \le i < j \le 9$, then let $A(i,j)$ be the determinant of $A(\boldsymbol{g})$ after columns i and j have been deleted. Let $\varphi(\boldsymbol{g})_1$, $\varphi(\boldsymbol{g})_2$ and $\varphi(\boldsymbol{g})_3$ be the 3×2 matrices

$$\varphi(\boldsymbol{g})_1 = \begin{bmatrix} A(1,4)y^2 - A(2,4)xy + A(3,4)x^2 & A(1,7)y^2 - A(2,7)xy + A(3,7)x^2 \\ - A(4,5)xy + A(4,6)x^2 & -A(4,7)y^2 + A(5,7)xy - A(6,7)x^2 \\ -A(4,7)y^2 + A(4,8)xy - A(4,9)x^2 & + A(7,8)xy - A(7,9)x^2 \end{bmatrix},$$

$$\varphi(\boldsymbol{g})_2 = \begin{bmatrix} + A(1,2)xy - A(1,3)x^2 & A(1,7)y^2 - A(2,7)xy + A(3,7)x^2 \\ A(1,4)y^2 - A(1,5)xy + A(1,6)x^2 & -A(4,7)y^2 + A(5,7)xy - A(6,7)x^2 \\ -A(1,7)y^2 + A(1,8)xy - A(1,9)x^2 & + A(7,8)xy - A(7,9)x^2 \end{bmatrix},$$

and

$$\varphi(\boldsymbol{g})_3 = \begin{bmatrix} + A(1,2)xy - A(1,3)x^2 & A(1,4)y^2 - A(2,4)xy + A(3,4)x^2 \\ A(1,4)y^2 - A(1,5)xy + A(1,6)x^2 & - A(4,5)xy + A(4,6)x^2 \\ -A(1,7)y^2 + A(1,8)xy - A(1,9)x^2 & -A(4,7)y^2 + A(4,8)xy - A(4,9)x^2 \end{bmatrix},$$

with entries from $k[x,y]_2$.

(e) Let $N(\boldsymbol{g})$ be the 2×3 Jacobian matrix

$$\begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial x} & \frac{\partial g_3}{\partial x} \\ \frac{\partial g_1}{\partial y} & \frac{\partial g_2}{\partial y} & \frac{\partial g_3}{\partial y} \end{bmatrix},$$

whose entries are homogeneous elements of degree three in $k[x,y]$.

(f) Let φ be a 3×2 matrix of homogeneous polynomials of degree 2 from the ring $k[x,y]$. Write $\varphi = \sum_{i=0}^{2} \varphi(i) x^i y^{2-i}$, where each $\varphi(i)$ is a 3×2 matrix of constants. Consider 2 new indeterminates w_1, w_2. Let $\boldsymbol{w}$ represent the vector

$$\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

and $p(\varphi)$ be the polynomial

$$\det \left[\varphi(0)\boldsymbol{w} \quad | \quad \varphi(1)\boldsymbol{w} \quad | \quad \varphi(2)\boldsymbol{w} \right]$$

of $k[w_1, w_2]$.

.D28.9 **Definition 8.6.** Adopt the notation of Definitions 8.4 and 8.5.

(1) Let $S_1 = \{\boldsymbol{g} \in \mathbb{A} \mid$ the ideal $I_{\boldsymbol{g}}$ of R has height at most one$\}$.
(2) Let $S_2 = \{\boldsymbol{g} \in \mathbb{A} \mid I_6(M(\boldsymbol{g})) = 0\}$.
(3) Let $S_3 = \{\boldsymbol{g} \in \mathbb{A} \mid \det(W(\boldsymbol{g})) = 0\}$.
(4) Let

$$S_4 = \left\{ \boldsymbol{g} \in \mathbb{A} \;\middle|\; \begin{array}{l} p(\varphi(\boldsymbol{g})_1),\ p(\varphi(\boldsymbol{g})_2), \text{ and } p(\varphi(\boldsymbol{g})_3) \text{ all are perfect cubes} \\ \text{in } k[w_1, w_2] \end{array} \right\}.$$

(5) Let

$$S_5 = \left\{ \boldsymbol{g} \in \mathbb{A} \,\middle|\, \begin{array}{l} p(\varphi(\boldsymbol{g})_1),\ p(\varphi(\boldsymbol{g})_2), \text{ and } p(\varphi(\boldsymbol{g})_3) \text{ each have at most} \\ \text{two distinct linear factors in } k[w_1, w_2] \end{array} \right\}.$$

(6) Define $\mathbb{U} \subseteq \mathbb{U}'$ to be the subsets

$$\mathbb{U} = \mathbb{A} \setminus (S_1 \cup S_2 \cup S_3) \subseteq \mathbb{U}' = \mathbb{A} \setminus (S_1 \cup S_2)$$

of $\mathbb{A}$.

(7) Define

$$C_i = \{\boldsymbol{g} \in \mathbb{A} \mid\ I_2(N(\boldsymbol{g})) \text{ has a common factor of degree } i\}.$$

Note. It is clear that $S_4 \subseteq S_5$ and that $C_3 \subseteq C_2 \subseteq C_1 \subseteq C_0 = \mathbb{A}$.

.XXX **Lemma 8.7.** *Retain the language of Definition 8.6. Each set C_i, and each of the sets S_i with $1 \le i \le 5$, is a closed subset of $\mathbb{A}$.*

Proof. Lemma ??? shows that S_1 is the closed subset $S_{\boldsymbol{d};1}(V)$ of $\mathbb{A} = \mathbb{A}_{\boldsymbol{d}}V$, where $\boldsymbol{d} = (4,4,4)$. It is obvious that S_2 and S_3 are closed subsets of $\mathbb{A}$. If p is a homogeneous cubic in $k[w_1, w_2]$, then

$$p \text{ is a perfect cube} \iff \operatorname{ht}\left(p, \frac{\partial p}{\partial w_1}, \frac{\partial p}{\partial w_2}, \frac{\partial^2 p}{\partial w_1^2}, \frac{\partial^2 p}{\partial w_1 \partial w_2}, \frac{\partial^2 p}{\partial w_2^2}\right) k[w_1, w_2] \le 1$$

and

$$p \text{ has at most two distinct linear factors} \iff \operatorname{ht}\left(p, \frac{\partial p}{\partial w_1}, \frac{\partial p}{\partial w_2}\right) k[w_1, w_2] \le 1.$$

Lemma ??? shows that the conditions on the right are closed conditions which are defined by the vanishing of certain polynomials in the coefficients of p. We conclude that S_4 and S_5 are also closed subsets of $\mathbb{A}$.

Now we consider the sets C_i of Definition 8.6. Keep in mind that every homogeneous polynomial in $k[x, y]$ factors into linear factors because k is algebraically closed. Thus, the generators of $I_2(N(\boldsymbol{g}))$ have a common factor of degree equal to i if and only if they have a common factor of degree at least i. The element $\boldsymbol{g}$ of $\mathbb{A}$ is in C_i if and only if the polynomials which define the closed set $S_{(6,6,6);i}$ of $\mathbb{A}_{(6,6,6)}$ vanish at the generators of $I_2(N(\boldsymbol{g}))$. It follows that each C_i is a closed subset of $\mathbb{A}$. $\square$

Remark. For S_4 and S_5 one could appeal to statements about the vanishing of certain resultants in place of our appeal to Lemma ???.

.Smry **Theorem 8.8.** *Adopt the language of Definition 8.6.*

(a) *If $\boldsymbol{g}$ is in $\mathbb{A}$, then the morphism $\Psi_{\boldsymbol{g}} \colon \mathbb{P}^1 \to \mathcal{C}_{\boldsymbol{g}}$ is birational and has no base points if and only if $\boldsymbol{g}$ is in the open subset $\mathbb{U}'$ of $\mathbb{A}$.*
(b) *If $\boldsymbol{g}$ is in the open subset $\mathbb{U}'$ of $\mathbb{A}$, then the curve $\mathcal{C}_{\boldsymbol{g}}$ has a visible singularity of multiplicity three if and only if $\boldsymbol{g}$ is in the closed subset S_3 of $\mathbb{A}$.*
(c) *If $\boldsymbol{g}$ is in the open subset $\mathbb{U}'$ of $\mathbb{A}$, then every singularity of the curve $\mathcal{C}_{\boldsymbol{g}}$ has multiplicity 2 if and only if $\boldsymbol{g}$ is in the open subset $\mathbb{U}$ of $\mathbb{A}$.*
(d) *If $\boldsymbol{g}$ is in the open subset $\mathbb{U}$ of $\mathbb{A}$, then the curve $\mathcal{C}_{\boldsymbol{g}}$ has exactly one visible singularity if and only if $\boldsymbol{g}$ is in the closed subset S_4 of $\mathbb{A}$.*
(e) *If $\boldsymbol{g}$ is in the open subset $\mathbb{U}$ of $\mathbb{A}$, then the curve $\mathcal{C}_{\boldsymbol{g}}$ has exactly two visible singularities if and only if $\boldsymbol{g}$ is in*

$$S_5 \cap (\mathbb{A} \setminus S_4).$$

Recall that S_5 is a closed subset of $\mathbb{A}$ and $(\mathbb{A} \setminus S_4)$ is an open subset of $\mathbb{A}$.
(f) *If $\boldsymbol{g}$ is in the open subset $\mathbb{U}$ of $\mathbb{A}$, then the curve $\mathcal{C}_{\boldsymbol{g}}$ has exactly three visible singularities if and only if $\boldsymbol{g}$ is in the open subset $(\mathbb{A} \setminus S_5)$ of $\mathbb{A}$.*
(g) *If the characteristic of k is zero, then the assertions of Table 8.2 hold.*

Proof. Assertion (a) is Lemma 8.10; (b) and (c) are Lemma 8.12; (d), (e), and (f) are Lemma 8.20; and (g) is Lemma 8.22. □

.'L27.23 **Lemma 8.9.** *Adopt the language of Definition 8.6. If $\boldsymbol{g} \in \mathbb{A}$, then the morphism $\Psi_{\boldsymbol{g}}$ is defined on all of $\mathbb{P}^1$ if and only if $\boldsymbol{g}$ is in the open set $\mathbb{A} \setminus S_1$ of $\mathbb{A}$.*

Proof. The field k is algebraically closed and therefore, the zero locus, $X(I_{\boldsymbol{g}})$, of $I_{\boldsymbol{g}}$ in $\mathbb{P}^1$ is non-empty if and only if the ideal $I_{\boldsymbol{g}}$ has height at most one. □

.'L27.22 **Lemma 8.10.** *Adopt the language of Definition 8.6. If $\boldsymbol{g}$ is in $\mathbb{A}$, then the morphism $\Psi_{\boldsymbol{g}} \colon \mathbb{P}^1 \to \mathcal{C}_{\boldsymbol{g}}$ is birational and has no base points if and only if $\boldsymbol{g}$ is in the open subset $\mathbb{U}'$ of $\mathbb{A}$.*

Proof. Fix an element $\boldsymbol{g} = (g_1, g_2, g_3)$ of $\mathbb{A}$. We know from Lemma 8.9 that $\Psi_{\boldsymbol{g}}$ is defined on all of $\mathbb{P}^1$ if and only if $\boldsymbol{g}$ is in the open subset $(\mathbb{A} \setminus S_1)$ of $\mathbb{A}$. The set $\mathbb{U}'$ is defined to be the intersection of the open subsets $(\mathbb{A} \setminus S_1) \cap (\mathbb{A} \setminus S_2)$. We assume that $\boldsymbol{g}$ is in $(\mathbb{A} \setminus S_1)$ and we prove that the morphism $\Psi_{\boldsymbol{g}}$ birational if and only if $\boldsymbol{g}$ is in $(\mathbb{A} \setminus S_2)$.

The ring $k[g_1, g_2, g_3]$ is the coordinate ring of the curve $\mathcal{C}_{\boldsymbol{g}}$. Furthermore, the ring homomorphism $k[T_1, T_2, T_3] \to k[g_1, g_2, g_3]$, which sends T_i to g_i, induces an isomorphism

$$\frac{k[T_1, T_2, T_3]}{(f)} \cong k[g_1, g_2, g_3],$$

where f is the defining equation of the curve $\mathcal{C}_{\boldsymbol{g}}$. The degree of f is

$$\begin{cases} 4 & \text{if } \Psi_{\boldsymbol{g}} \text{ is a birational morphism} \\ 2 \text{ or } 1 & \text{if } \Psi_{\boldsymbol{g}} \text{ is not a birational morphism.} \end{cases}$$

Thus, the common value of

$$\dim \left(\frac{k[T_1, T_2, T_3]}{(f)} \right)_2 = \dim k[g_1, g_2, g_3]_8$$

is

$$\begin{cases} \text{equal to } 6 & \text{if } \Psi_{\boldsymbol{g}} \text{ is a birational morphism} \\ \text{less than } 6 & \text{if } \Psi_{\boldsymbol{g}} \text{ is not a birational morphism.} \end{cases}$$

On the other hand, the dimension of $k[g_1, g_2, g_3]_8$ is equal to the dimension of the vector space generated by $g_1^2, g_1g_2, g_1g_3, g_2^2, g_2g_3, g_3^2$. Thus,

$$\Psi_{\boldsymbol{g}} \text{ is not birational} \iff \dim k[g_1, g_2, g_3]_8 < 6$$

$$\iff \operatorname{rank} M(\boldsymbol{g}) < 6 \iff I_6(M(\boldsymbol{g})) = 0 \iff \boldsymbol{g} \in S_2. \quad \square$$

.'O27.8 **Observation 8.11.** *Adopt the notation of Definition 8.5. Let $\boldsymbol{g}$ be an element of $\mathbb{A}$. Then there is a non-trivial linear relation on the row vector $G(\boldsymbol{g})$ if and only if $\det W(\boldsymbol{g}) = 0$.*

Proof. Let $\rho^{(5)}$ and $N^{(1)}$ be the matricies

$$\rho^{(5)} = [\, y^5 \quad xy^4 \quad x^2y^3 \quad x^3y^2 \quad x^4y \quad x^5 \,] \quad \text{and} \quad N^{(1)} = \begin{bmatrix} y & x & 0 & 0 & 0 & 0 \\ 0 & 0 & y & x & 0 & 0 \\ 0 & 0 & 0 & 0 & y & x \end{bmatrix}.$$

Matrix multiplication gives

$$\rho^{(5)} W(\boldsymbol{g}) = [\, yg_1 \quad xg_1 \quad yg_2 \quad xg_2 \quad yg_3 \quad xg_3 \,] = G(\boldsymbol{g}) N^{(1)}.$$

If

$$\boldsymbol{\ell} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix}$$

is a matrix of linear forms from $k[x,y]$, then $\ell_j = a_{0,j}y + a_{1,j}x$ for some constants $a_{i,j}$ in k. It follows that $\boldsymbol{\ell} = N^{(1)}\boldsymbol{a}$, where $\boldsymbol{a}$ is the following 6×1 matrix of constants:

$$\boldsymbol{a} = \begin{bmatrix} a_{0,1} \\ a_{1,1} \\ a_{0,2} \\ a_{1,2} \\ a_{0,3} \\ a_{1,3} \end{bmatrix}.$$

We see that

$$G(\boldsymbol{g})\boldsymbol{\ell} = G(\boldsymbol{g})N^{(1)}\boldsymbol{a} = \rho^{(5)}W(\boldsymbol{g})\boldsymbol{a}.$$

A polynomial is the zero polynomial only if all of the coefficients are zero; hence, the polynomial $\rho^{(5)}W(\boldsymbol{g})\boldsymbol{a}$ is the zero polynomial only if $W(\boldsymbol{g})\boldsymbol{a}$ is the zero vector. We now see that

$$\begin{array}{ll} & \text{there exists a non-zero } 3\times 1 \text{ matrix } \boldsymbol{\ell} \text{ of linear forms with } G(\boldsymbol{g})\boldsymbol{\ell} = 0 \\ \iff & \text{there exists a non-zero } 6\times 1 \text{ matrix } \boldsymbol{a} \text{ of constants with } W(\boldsymbol{g})\boldsymbol{a} = 0 \\ \iff & \det W(\boldsymbol{g}) = 0. \quad \square \end{array}$$

.'L27.24 **Lemma 8.12.** *Adopt the language of Definition 8.6. Let $\boldsymbol{g}$ be in the open subset $\mathbb{U}'$ of $\mathbb{A}$. The following statements hold.*

(a) *The curve $\mathcal{C}_{\boldsymbol{g}}$ has a visible singularity of multiplicity three if and only if $\boldsymbol{g}$ is in the closed subset S_3 of $\mathbb{A}$.*

(b) *Every singularity of the curve $\mathcal{C}_{\boldsymbol{g}}$ has multiplicity 2 if and only if $\boldsymbol{g}$ is in the open subset $\mathbb{U}$ of $\mathbb{A}$.*

Proof. Fix $\boldsymbol{g} = (g_1, g_2, g_3) \in \mathbb{U}'$. It follows that the ideal $I_{\boldsymbol{g}}$ has height two. According to the Hilbert-Burch theorem, the resolution of $I_{\boldsymbol{g}}$ has the form

$$0 \to R(-4-d_1) \oplus R(-4-d_2) \to R(-4)^3 \to I_{\boldsymbol{g}},$$

where (d_1, d_2) is equal to either $(1,3)$ or $(2,2)$. (The hypothesis ensures that the morphism $\Psi_{\boldsymbol{g}}$ is birational; so, $(d_1,d_2) = (0,4)$ is not possible.) The curve $\mathcal{C}_{\boldsymbol{g}}$ has degree 4; hence, the singularities of $\mathcal{C}_{\boldsymbol{g}}$ have multiplicity two or three. Corollary 1.23 shows that $\mathcal{C}_{\boldsymbol{g}}$ has a visible singularity of multiplicity three if and only if $(d_1, d_2) = (1,3)$. Thus,

$$\begin{array}{ll} & \mathcal{C}_{\boldsymbol{g}} \text{ has a visible singularity of multiplicity three} \\ \iff & \text{there is a non-trivial linear relation on the row vector } G(\boldsymbol{g}) \\ \iff & \det W(\boldsymbol{g}) = 0 \iff \boldsymbol{g} \in S_3. \end{array}$$

Observation 8.11 gives the middle equivalence. $\square$

.'O27.9 **Observation 8.13.** *Adopt the notation of Definition 8.5. Let* $\boldsymbol{g}$ *be an element of* $\mathbb{U}$. *Then the following statements hold.*

(1) *At least one of the constants* $c_{0,1}$, $c_{0,2}$, *or* $c_{0,3}$ *is non-zero.*
(2) *If* $c_{0,i} \neq 0$, *then* $\varphi(\boldsymbol{g})_i$ *is a Hilbert-Burch matrix for* $I_{\boldsymbol{g}}$.
(3) *If* $c_{0,i} = 0$, *then the columns of the matrix* $\varphi(\boldsymbol{g})_i$ *are linearly dependent over* k.

Proof. The hypothesis $\boldsymbol{g} \in \mathbb{U}$, ensures that $\operatorname{ht}(I_{\boldsymbol{g}}) = 2$; thus $I_{\boldsymbol{g}}$ is not contained in (x) and it is not possible for $c_{0,1}$, $c_{0,2}$, and $c_{0,3}$ to all be zero. Assertion (1) is established.

We next demonstrate that the row vector $G(\boldsymbol{g})$ times the 3×2 matrix $\varphi(\boldsymbol{g})_i$ is equal to zero for $1 \leq i \leq 3$. Let $\rho^{(6)}$ and $N^{(2)}$ be the matrices

$$\rho^{(6)} = [\, y^6 \quad xy^5 \quad x^2y^4 \quad x^3y^3 \quad x^4y^2 \quad x^5y \quad x^6 \,] \quad \text{and}$$

$$N^{(2)} = \begin{bmatrix} y^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^2 & xy & y^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & y^2 & xy & y^2 \end{bmatrix}.$$

Matrix multiplication gives

$$\begin{aligned} \rho^{(6)} A(\boldsymbol{g}) &= [\, y^2 g_1 \quad xyg_1 \quad x^2 g_1 \quad y^2 g_2 \quad xyg_2 \quad x^2 g_2 \quad y^2 g_3 \quad xyg_3 \quad x^2 g_3 \,] \\ &= G(\boldsymbol{g}) N^{(2)}. \end{aligned}$$

If

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

is a matrix of quadratic forms from $k[x,y]$, then $q_j = b_{0,j}y^2 + b_{1,j}xy + b_{1,j}x^2$ for some constants $b_{i,j}$ in k. It follows that $\boldsymbol{q} = N^{(2)}\boldsymbol{b}$, where $\boldsymbol{b}$ is the following 9×1 matrix of constants:

$$\boldsymbol{b} = \begin{bmatrix} b_{0,1} \\ b_{1,1} \\ b_{2,1} \\ b_{0,2} \\ b_{1,2} \\ b_{2,2} \\ b_{0,3} \\ b_{1,3} \\ b_{2,3} \end{bmatrix}.$$

We see that

.'gN
$$G(\boldsymbol{g})\boldsymbol{q} = G(\boldsymbol{g})N^{(2)}\boldsymbol{b} = \rho^{(6)}A(\boldsymbol{g})\boldsymbol{b}. \tag{8.14}$$

One may obtain nine "Eagon-Northcott" relations on $A(\boldsymbol{g})$, by crossing one column of $A(\boldsymbol{g})$ at a time and computing the signed maximal minors of the resulting 7×8 matrix. In particular, when one crosses out columns 1, 4, or 7 of $A(\boldsymbol{g})$, one obtains the relations

$$\boldsymbol{b}^{(1)} = \begin{bmatrix} 0 \\ A(1,2) \\ -A(1,3) \\ A(1,4) \\ -A(1,5) \\ A(1,6) \\ -A(1,7) \\ A(1,8) \\ -A(1,9) \end{bmatrix}, \quad \boldsymbol{b}^{(4)} = \begin{bmatrix} A(1,4) \\ -A(2,4) \\ A(3,4) \\ 0 \\ -A(4,5) \\ A(4,6) \\ -A(4,7) \\ A(4,8) \\ -A(4,9) \end{bmatrix}, \quad \text{and} \quad \boldsymbol{b}^{(7)} = \begin{bmatrix} A(1,7) \\ -A(2,7) \\ A(3,7) \\ -A(4,7) \\ A(5,7) \\ -A(6,7) \\ 0 \\ A(7,8) \\ -A(7,9) \end{bmatrix}$$

on $A(\boldsymbol{g})$, respectively. It follows from (8.14), that the corresponding relations on $G(\boldsymbol{g})$ are

$$\begin{aligned} N^{(2)}\boldsymbol{b}^{(1)} &= (\varphi(\boldsymbol{g})_2)_{*,1} = (\varphi(\boldsymbol{g})_3)_{*,1}, \\ N^{(2)}\boldsymbol{b}^{(4)} &= (\varphi(\boldsymbol{g})_1)_{*,1} = (\varphi(\boldsymbol{g})_3)_{*,2}, \quad \text{and} \\ N^{(2)}\boldsymbol{b}^{(7)} &= (\varphi(\boldsymbol{g})_1)_{*,2} = (\varphi(\boldsymbol{g})_2)_{*,2}, \end{aligned} \tag{4.4}$$

where $M_{*,j}$ denotes the j^{th} column of M. Therefore, $G(\boldsymbol{g})\varphi(\boldsymbol{g})_i = 0$ for $1 \le i \le 3$.

The ideal $I_{\boldsymbol{g}}$ of $R = k[x,y]$ has height two. The hypothesis $\boldsymbol{g} \in \mathbb{U}$ ensures $\boldsymbol{g} \notin S_3$. Thus, $\det W(\boldsymbol{g}) \neq 0$ and there are no linear relations on $G(\boldsymbol{g})$. It follows, in particular, that g_1, g_2, and g_3 are minimal generators of $I_{\boldsymbol{g}}$. The Hilbert-Burch Theorem guarantees that the minimal resolution of $R/I_{\boldsymbol{g}}$ looks like

$$0 \to R(-8)^2 \xrightarrow{\varphi} R(-6)^3 \xrightarrow{G(\boldsymbol{g})} R.$$

To identify φ, we need only find two linearly independent quadratic relations on $G(\boldsymbol{g})$. Expand the minors $A(4,7)$, $A(1,7)$, and $A(1,4)$ of A across the first row to see that

$$A(4,7) = c_{0,1}\det W(\boldsymbol{g}), \quad A(1,7) = c_{0,2}\det W(\boldsymbol{g}), \quad \text{and} \quad A_{1,4} = c_{0,3}\det W(\boldsymbol{g}).$$

The determinant of $W(\boldsymbol{g})$ is not zero by hypothesis; so $A(4,7)$ is not zero whenever $c_{0,1} \neq 0$; $A(1,7) \neq 0$ whenever $c_{0,2} \neq 0$; and $A(1,4) \neq 0$ whenever $c_{0,3} \neq 0$. A quick look at $\varphi(\boldsymbol{g})_i$ shows that the columns of $\varphi(\boldsymbol{g})_i$ are linearly independent whenever $c_{0,i} \neq 0$, for $1 \leq i \leq 3$. Assertion (2) is established.

(3) Once again we consider the Eagon-Northcott complex associated to the map $A(\boldsymbol{g})\colon R^9 \to R^7$:

$$0 \to R^7 \otimes \bigwedge^9 R^9 \to \bigwedge^8 R^9 \to R^9 \xrightarrow{A(\boldsymbol{g})} R^7.$$

Each row of $A(\boldsymbol{g})$ gives rise to a relation on the nine Eagon-Northcott relations on $A(\boldsymbol{g})$. In particular, the top row of $A(\boldsymbol{g})$ gives rise to the relation

$$c_{0,1}\boldsymbol{b}^{(1)} - c_{0,2}\boldsymbol{b}^{(4)} + c_{0,3}\boldsymbol{b}^{(7)} = 0.$$

Use (4.4) to see that

$$\begin{cases} [\varphi(\boldsymbol{g})_1] \begin{bmatrix} -c_{0,2} \\ c_{0,3} \end{bmatrix} = 0 \quad \text{if } c_{0,1} = 0 \\ [\varphi(\boldsymbol{g})_2] \begin{bmatrix} +c_{0,1} \\ c_{0,3} \end{bmatrix} = 0 \quad \text{if } c_{0,2} = 0 \\ [\varphi(\boldsymbol{g})_3] \begin{bmatrix} c_{0,1} \\ -c_{0,2} \end{bmatrix} = 0 \quad \text{if } c_{0,3} = 0. \end{cases}$$

Assertion (1) guarantees that the above relations are non-trivial. $\square$

.p653 **Lemma 8.15.** *Let φ be a matrix as described in Definition 8.5 (f). If the columns of φ are linearly dependent over k, then $p(\varphi)$ is a perfect cube in $k[w_1, w_2]$.*

Proof. We are told that there exists a non-zero vector $\left[\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right]$ with entries from k so that $\varphi \left[\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right] = 0$. We demonstrate that $c_1^3 p(\varphi)$ is a perfect cube. A slight modification of our argument shows that $c_2^3 p(\varphi)$ is a perfect cube. Since at least one of the c_i is a unit, we conclude that $p(\varphi)$ is a perfect cube. Write $\varphi = \sum \varphi(i) x^i y^{2-i}$ as described in Definition 8.5. We have $\varphi(i) \left[\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right] = 0$; so,

$$\begin{aligned} c_1\varphi(i) \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] &= w_1 c_1 \varphi(i)_{*,1} + w_2 c_1 \varphi(i)_{*,2} = -w_1 c_2 \varphi(i)_{*,2} + w_2 c_1 \varphi(i)_{*,2} \\ &= (w_2 c_1 - w_1 c_2)\varphi(i)_{*,2}, \end{aligned}$$

where $\varphi(i)_{*,j}$ is column j of $\varphi(i)$. Let $\lambda = c_1 w_2 - c_2 w_1$. It follows that

$$\begin{aligned} c_1^3 p(\varphi) &= \det [\, c_1\varphi(0)\boldsymbol{w} \mid c_1\varphi(1)\boldsymbol{w} \mid c_1\varphi(2)\boldsymbol{w} \,] \\ &= \det [\, \lambda\varphi(0)_{*,2} \mid \lambda\varphi(1)_{*,2} \mid \lambda\varphi(2)_{*,2} \,] \\ &= \lambda^3 \det [\, \varphi(0)_{*,2} \mid \varphi(1)_{*,2} \mid \varphi(2)_{*,2} \,], \end{aligned}$$

and $c_1^3 p_1(M)$ is a perfect cube in $k[w_1, w_2]$ as claimed. $\square$

L30.1 **Lemma 8.16.** *Adopt the language of Definition 8.6. Let $\boldsymbol{g}$ be an element of the open subset $\mathbb{U}$ of $\mathbb{A}$ and let φ be any homogeneous Hilbert-Burch matrix for the ideal $I_{\boldsymbol{g}}$. Then the number of visible singularities of $\mathcal{C}_{\boldsymbol{g}}$ is equal to the number of distinct linear factors of $p(\varphi)$.*

Proof. Recall, from Observation 8.13, that every entry of φ is a quadratic form. Consider the following five sets:

$$\begin{aligned}
X_1 &= \{u \in \mathbb{P}^2 \mid u \text{ is a singular point on } \mathcal{C}_{\boldsymbol{g}}\},\\
X_2 &= \{u \in \mathbb{P}^2 \mid \dim I_1(u\varphi) \le 1\},\\
X_3 &= \{(u,w) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid u\varphi w = 0\},\\
X_4 &= \{w \in \mathbb{P}^1 \mid \dim I_1(\varphi w) \le 2\},\\
X_5 &= \{w \in \mathbb{P}^1 \mid w \text{ is a root of } p(\varphi)\}.
\end{aligned}$$

In this discussion u in $\mathbb{P}^2$ is represented by a row vector $[u_1, u_2, u_3]$ and w in $\mathbb{P}^1$ is represented by a column vector $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$. Observe that the defining conditions for the sets X_i are unperturbed if $[u_1, u_2, u_3]$ is replaced by $\alpha[u_1, u_2, u_3]$ or $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$ is replaced by $\alpha \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$ for any non-zero scalar α. We prove the result by establishing

(1) $X_4 = X_5$,
(2) the projection map $X_3 \to X_4$ is a bijection,
(3) the projection map $X_3 \to X_2$ is a bijection, and
(4) $X_1 = X_2$.

We begin by making the critical matrix calculation. Use the notation of Definition 8.5 (f). Fix $w \in \mathbb{P}^1$. Observe that

$$\begin{aligned}
\varphi w &= [y^2\varphi(0) + xy\varphi(1) + x^2\varphi(2)]w = [\,\varphi(0) \mid \varphi(1) \mid \varphi(2)\,] \begin{bmatrix} y^2 & 0 \\ 0 & y^2 \\ xy & 0 \\ 0 & xy \\ x^2 & 0 \\ 0 & x^2 \end{bmatrix} w \\
&= [\,\varphi(0) \mid \varphi(1) \mid \varphi(2)\,] \begin{bmatrix} w & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} y^2 \\ xy \\ x^2 \end{bmatrix} \\
&= [\,\varphi(0)w \mid \varphi(1)w \mid \varphi(2)w\,] \begin{bmatrix} y^2 \\ xy \\ x^2 \end{bmatrix}
\end{aligned}$$

We prove (1). Fix $w \in \mathbb{P}^1$. Observe that

$$\begin{aligned} w \in X_5 &\iff \det[\,\varphi(0)w \quad | \quad \varphi(1)w \quad | \quad \varphi(2)w\,] = 0 \\ &\iff \text{the matrix } [\,\varphi(0)w \quad | \quad \varphi(1)w \quad | \quad \varphi(2)w\,] \text{ is singular} \\ &\iff \exists u \in \mathbb{P}^1 \text{ with } u\,[\,\varphi(0)w \quad | \quad \varphi(1)w \quad | \quad \varphi(2)w\,] = 0 \\ &\iff \exists u \in \mathbb{P}^1 \text{ with } u\,[\,\varphi(0)w \quad | \quad \varphi(1)w \quad | \quad \varphi(2)w\,] \begin{bmatrix} y^2 \\ xy \\ x^2 \end{bmatrix} = 0 \\ &\iff \exists u \in \mathbb{P}^1 \text{ with } u\varphi w = 0 \iff \dim I_1(\varphi w) \le 2 \iff w \in X_4. \end{aligned}$$

30.2

We prove (2) and (3). The hypothesis that $\operatorname{ht} I_2(\varphi) = 2$ ensures that

$$w \in \mathbb{P}^1 \implies |\{u \in \mathbb{P}^2 \mid u\varphi w = 0\}| \le 1 \quad \text{and} \tag{8.17}$$

$$u \in \mathbb{P}^2 \implies |\{w \in \mathbb{P}^1 \mid u\varphi w = 0\}| \le 1. \tag{8.18}$$

Indeed, if $w \in \mathbb{P}^1$, then the matrix φw is a 3×1 matrix of quadratic forms from $k[x,y]$. Observe that $\dim I_1(\varphi w) \ge 2$. Otherwise $I_1(\varphi w)$ is a principal ideal and $I_2(\varphi)$ is contained in a height one ideal. We conclude that the left null space of φw has dimension at most 1 and (8.17) is established. Take $u \in \mathbb{P}^2$. The matrix $u\varphi$ is a 1×2 matrix of quadratic forms from $k[x,y]$. The matrix $u\varphi$ can not be identically zero; otherwise $I_2(\varphi)$ is a principal ideal. We conclude that the right null space of $u\varphi$ has dimension at most 1 and (8.18) is established.

If $w \in X_4$, then (8.17) guarantees that there is exactly one $u \in \mathbb{P}^2$ with (u,w) in X_3. Similarly, if $u \in X_2$, then (8.18) guarantees that there is exactly one $w \in \mathbb{P}^1$ with $(u,w) \in X_3$.

We prove (4). General Lemma 1.7 (together with Remark 1.20) asserts that

$$u \in X_1 \iff \deg(\gcd I_1(u\varphi)) \ge 2.$$

On the other hand, the entries of $u\varphi$ have degree 2; so,

$$u \in X_1 \iff \text{the entries of } u\varphi \text{ are linearly dependent} \iff u \in X_2. \quad \square$$

.'C27' **Corollary 8.19.** *Adopt the language of Definition 8.6. Let $\boldsymbol{g}$ be an element of the open subset $\mathbb{U}$ of $\mathbb{A}$.*

(1) *If $c_{0,i} \neq 0$, then the number of visible singularities of $\mathcal{C}_{\boldsymbol{g}}$ is equal to the number of distinct linear factors of $p(\varphi(\boldsymbol{g})_i)$.*
(2) *If $c_{0,i} = 0$, then $p(\varphi(\boldsymbol{g})_i)$ is a perfect cube in $k[w_1, w_2]$.*

Proof. Apply Lemmas 8.16 and 8.15 together with Observation 8.13. $\square$

.'L27.26 **Lemma 8.20.** *Adopt the language of Definition 8.6. Let* $\boldsymbol{g}$ *be in the open subset* $\mathbb{U}$ *of* $\mathbb{A}$*.*

(a) *The curve* $\mathcal{C}_{\boldsymbol{g}}$ *has exactly one visible singularity if and only if* $\boldsymbol{g}$ *is in the closed subset* S_4 *of* $\mathbb{A}$*.*

(b) *The curve* $\mathcal{C}_{\boldsymbol{g}}$ *has exactly two visible singularities if and only if* $\boldsymbol{g}$ *is in*

$$S_5 \cap (\mathbb{A} \setminus S_4).$$

Recall that S_5 *is a closed subset of* $\mathbb{A}$ *and* $(\mathbb{A} \setminus S_4)$ *is and open subset of* $\mathbb{A}$*.*

(c) *The curve* $\mathcal{C}_{\boldsymbol{g}}$ *has exactly three visible singularities if and only if* $\boldsymbol{g}$ *is in the open subset* $\mathbb{A} \setminus S_5$ *of* $\mathbb{A}$*.*

Proof. Recall from Observation 8.13 that the hypothesis $\boldsymbol{g} \in \mathbb{U}$ guarantees that some coefficient $c_{0,i}$ is non-zero. Thus, Proposition 8.19 shows that the number of visible singularities on $\mathcal{C}_{\boldsymbol{g}}$ is equal to

num
$$\max\{\text{the number of distinct linear factors of } p(\varphi(\boldsymbol{g})_i) \mid 1 \le i \le 3\}. \tag{8.21}$$

The definition of the sets S_4 and S_5 yields that

$$\begin{array}{lll} \boldsymbol{g} \in S_4 & \iff & \text{the number of (8.21) is 1, and} \\ \boldsymbol{g} \in S_5 & \iff & \text{the number of (8.21) is } \le 2. \quad \square \end{array}$$

lst **Lemma 8.22.** *Adopt the language of Definition 8.6. Assume that the field* k *has characteristic zero. The assertions of Table 8.2 hold.*

Proof. We have already established (8.1). It remains to examine how the number of branchs at each singularity p of $\mathcal{C}_{\boldsymbol{g}}$ is related to the value of i with $\boldsymbol{g} \in C_i \setminus C_{i+1}$.

Suppose first that $\boldsymbol{g} \in \mathrm{VS}_3(1)$. In this case $\mathcal{C}_{\boldsymbol{g}}$ has exactly one singularity, which we call p. Furthermore, we know, from Theorem 0.2 (2), that $\deg \gcd I_2(N(\boldsymbol{g})) = m_p - s_p = 3 - s_p$. Thus, if $\boldsymbol{g} \in C_i \setminus C_{i+1}$, then $s_p = 3 - i$; that is,

$$\begin{array}{lll} \boldsymbol{g} \in \mathrm{VS}_3(1) \cap C_2 & \Longrightarrow & s_p = 1 \\ \boldsymbol{g} \in \mathrm{VS}_3(1) \cap C_1 \setminus C_2 & \Longrightarrow & s_p = 2 \\ \boldsymbol{g} \in \mathrm{VS}_3(1) \setminus C_1 & \Longrightarrow & s_p = 3. \end{array}$$

Now we suppose that $\boldsymbol{g} \in \mathrm{VS}_2(1)$. Once again, $\mathcal{C}_{\boldsymbol{g}}$ has exactly one singularity, which we call p. This time, $\deg \gcd I_2(N(\boldsymbol{g})) = m_p - s_p = 2 - s_p$. Thus, if $\boldsymbol{g} \in C_i \setminus C_{i+1}$, then $s_p = 2 - i$; that is,

$$\begin{array}{lll} \boldsymbol{g} \in \mathrm{VS}_2(1) \cap C_1 & \Longrightarrow & s_p = 1 \\ \boldsymbol{g} \in \mathrm{VS}_3(1) \setminus C_1 & \Longrightarrow & s_p = 2. \end{array}$$

If $\boldsymbol{g} \in \mathrm{VS}(2)$, then $\mathcal{C}_{\boldsymbol{g}}$ has two singularities: p_1 and p_2. We have

$$\deg \gcd I_2(N(\boldsymbol{g})) = m_{p_1} + m_{p_2} - s_{p_1} - s_{p_2} = 4 - s_{p_1} - s_{p_2}.$$

Thus,

$$\begin{array}{lcl} \boldsymbol{g} \in \mathrm{VS}(2) \cap C_2 & \Longrightarrow & s_{p_1} = s_{p_2} = 1 \\ \boldsymbol{g} \in \mathrm{VS}(2) \cap C_1 \setminus C_2 & \Longrightarrow & \{s_{p_1}, s_{p_2}\} = \{1, 2\}. \\ \boldsymbol{g} \in \mathrm{VS}(2) \setminus C_1 & \Longrightarrow & s_{p_1} = s_{p_2} = 2. \end{array}$$

If $\boldsymbol{g} \in \mathrm{VS}(3)$, then $\mathcal{C}_{\boldsymbol{g}}$ has three singularities: p_1, p_2, p_3. We have

$$s_{p_1} + s_{p_2} + s_{p_3} = 6 - \deg \gcd I_2(N(\boldsymbol{g})).$$

Thus,

$$\begin{array}{lcl} \boldsymbol{g} \in \mathrm{VS}(3) \cap C_3 & \Longrightarrow & \text{all three } s_{p_i} \text{ equal 1,} \\ \boldsymbol{g} \in \mathrm{VS}(3) \cap C_2 \setminus C_3 & \Longrightarrow & \text{exactly two of the } s_{p_i} \text{ equal 1, the other one is 2,} \\ \boldsymbol{g} \in \mathrm{VS}(3) \cap C_1 \setminus C_2 & \Longrightarrow & \text{exactly one of the } s_{p_i} \text{ equals 1, the other two are 2,} \\ \boldsymbol{g} \in \mathrm{VS}(3) \setminus C_1 & \Longrightarrow & \text{all three } s_{p_i} \text{ equal 2.} \quad \square \end{array}$$

Section 9. The correspondence between the Hilbert-Burch matrices and the singularities of a rational quartic plane curve.

Section 10. Singularities of multiplicity degree divided by two.

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To Do List as of January 8, 2010 with respect to the SVRI paper

1. It would be nice to separate the one ambiguous case from Table 8.2.
2. What are the dimensions of these various closed sets?
3. Which closed sets are irreducible?
4. How does this work compare with

Bruce, J. W. and Giblin, P. J., *A stratification of the space of plane quartic curves*, Proc. London Math. Soc. (3) **42** (1981), no. 2, 270–298.

5. What other existing papers should we compare it to? In particular, we should say how the general lemma generalizes Song-Chen-Goldman.

N. Song, F. Chen, and R. Goldman, *Axial moving lines and singularities of rational planar curves*, Comput. Aided Geom. Design **24** (2007), no. 4, 200–209.

6. What should we include about the parameterizations? This will require a comparison with CTC Wall's paper

C. T. C. Wall, *Geometry of Quartic Curves*, Proc. Cambridge Phil. Soc. **117** (1995), 415-423.

   If we include this, we need to write a careful version of the final results about parameterizations. We also need to write down the triple lemma and the generalized triple lemma.
7. I think that Bernd would prefer that the ambient parameter space be a Grassmannian rather than an affine space. (Some of the ideas from the present draft also make sense in the Grassmannian context and we are able to compute the dimension of the non-birational locus there.)
8. I think I read that in some sense there is only one 3-cusp quartic. We should make sense of this.
9. Other references need to be added.
10. Section 3 is pretty long. Maybe it should probably be reconfigured into smaller sections. In particular, maybe the calculation $e(\hat{R}/J_i) = e(\widehat{\mathcal{O}_{\mathcal{C},p}}/\mathcal{J}_i)$ (see 1.19) should be made into a stand alone result.
11. Are either of the assertions of Corollary 7.3 well known? If so, we should point out that Theorem 7.2 generalizes known results.
12. Can we avoid characteristic zero in section 4? Can we avoid charactereistic zero in our decomposition into open and closed sets?
13. I just found Claudia's phrase: "based on Eisenbud-Ulrich interpretation of the fiber in terms of generalized rows of the presentation matrix" as as warm-up for the general lemma. Made we should work this into the paper.
14. Do we want to hype Theorem 7.2 in the introduction? The Jacobian matrix identifies the non-smooth branches as well as the multiplicity along each branch.
15. Do we want to split off an entire section called "A parameterization of the branches of a paramterized curve". It would include Remark 1.30, Observation 1.24, and most of the stuff between "We translate . . . " on page 31 until the end of the proof on page 34.
16. Emphasize in the introduction that most of the important results work in much more generality than plane quartic curves. Feel free to hint that we plan to

attack other situations.

17. The Song-Chen-Goldman result about no singularities between low degree and high degree is an immediate consequence of the triple lemma. This MUST be observed in the paper. No singularities of multiplicity more than d_2.

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