Noether normalization lemma

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In mathematics, the **Noether normalization lemma** is a result of commutative algebra, introduced by to Emmy Noether in 1926.[1] A simple version states that for any field k, and any finitely generated commutative k-algebra A, there exists a nonnegative integer d and algebraically independent elements y_1, y_2, ..., y_d in A such that A is a finitely generated module over the polynomial ring $S := k[y_1, y_2, ..., y_d]$.

The integer d is the Krull dimension of A (since A and S have the same dimension.) When A is an integral domain, d is the transcendence degree of the field of fractions of A over k.

The theorem has geometric interpretation. Suppose A is integral. Let S be the coordinate ring of d-dimensional affine space $\mathbb{A}_k^d$, and A as the coordinate ring of some other d-dimensional affine variety X. Then the inclusion map $S \to A$ induces a surjective finite morphism of affine varieties $X \to \mathbb{A}_k^d$. The conclusion is that any affine variety is a branched covering of affine space. When k is infinite, such a branched covering map can be constructed by taking a general projection from an affine space containing X to a d-dimensional subspace.

More generally, in the language of schemes, the theorem can equivalently be stated as follows: every affine k-scheme (of finite type) X is finite over an affine n-dimensional space. The theorem can be refined to include a chain of prime ideals of R (equivalently, irreducible subsets of X) that are finite over the affine coordinate subspaces of the appropriate dimensions.[2]

The form of the Noether normalization lemma stated above can

be used as an important step in proving Hilbert's Nullstellensatz. This gives it further geometric importance, at least formally, as the Nullstellensatz underlies the development of much of classical algebraic geometry. The theorem is also an important tool in establishing the notions of Krull dimension for k-algebras.

Proof

The following proof is due to Nagata and is taken from Mumford's red book. A proof in the geometric flavor is also given in the page 127 of the red book and this mathoverflow thread (http://mathoverflow.net/questions/42275/choosing-the-algebraic-independent-elements-in-noethers-normalization-lemma/42363#42363).

The ring A in the lemma is generated as k-algebra by elements, say, $y_1, ..., y_m$ such that $y_1, ..., y_d$ (some d) are algebraically independent over k and the rest are algebraic over $k[y_1, ..., y_d]$.[3] We shall induct on m. If $m = d$, then the assertion is trivial. Assume now $m > d$. It is enough to show that there is a subring S of A that is generated by $m - 1$ elements and is such that A is finite over S, for, by inductive hypothesis, we can find algebraically independent elements $x_1, ..., x_d$ of S such that S is finite over $k[x_1, ..., x_d]$. Since $m > d$, there is a nonzero polynomial f in m variables over k such that

$$f(y_1, ..., y_m) = 0.$$

Given an integer r which is determined later, set

$$z_i = y_i - y_1^{r^{i-1}}, \quad 2 \le i \le m.$$

Then the preceding reads:

$$f(y_1, z_2 + y_1^r, z_3 + y_1^{r^2}, ..., z_m + y_1^{r^{m-1}}) = 0.$$

Now, the highest term in y_1 of $ay_1^{\alpha_1} \prod_{2}^{m}(z_i + y_1^{r^{i-1}})^{\alpha_i}, a \in k,$ looks

$$ay_1^{\alpha_1 + r\alpha_2 + \ldots + \alpha_m r^{m-1}}.$$

Thus, if r is larger than any exponent α_i appearing in f, then the highest term of $f(y_1, z_2 + y_1^r, z_3 + y_1^{r^2}, \ldots, z_m + y_1^{r^{m-1}})$ in y_1 also has the form as above. In other words, y_1 is integral over $S = k[z_2, \ldots, z_m]$. Since $y_i = z_i + y_1^{r^{i-1}}$ are also integral over that ring, A is integral over S. It follows A is finite over S.

If A is an integral domain, then d is the transcendence degree of its field of fractions. Indeed, A and $S = k[y_1, \ldots, y_d]$ have the same transcendence degree (i.e., the degree of the field of fractions) since the field of fractions of A is algebraic over that of S (as A is integral over S) and S obviously has transcendence degree d. Thus, it remains to show the Krull dimension of the polynomial ring S is d. (this is also a consequence of dimension theory.) We induct on d, $d = 0$ being trivial. Since $0 \subsetneq (y_1) \subsetneq (y_1, y_2) \subsetneq \cdots \subsetneq (y_1, \ldots, y_d)$ is a chain of prime ideals, the dimension is at least d. To get the reverse estimate, let $0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_m$ be a chain of prime ideals. Let $0 \neq u \in \mathfrak{p}_1$. We apply the noether normalization and get $T = k[u, z_2, \ldots, z_d]$ (in the normalization process, we're free to choose the first variable) such that S is integral over T. By inductive hypothesis, $T/(u)$ has dimension d - 1. By incomparability, $\mathfrak{p}_i \cap T$ is a chain of length m and then, in $T/(\mathfrak{p}_1 \cap T)$, it becomes a chain of length $m - 1$. Since $\dim T/(\mathfrak{p}_1 \cap T) \leq \dim T/(u)$, we have $m - 1 \leq d - 1$. Hence, $\dim S \leq d$.

Notes

1. ^ Noether 1926
2. ^ Eisenbud 1995, Theorem 13.3
3. ^ cf. Exercise 16 in Ch. 5 of Atiyah-MacDonald, Introduction to commutative algebra.

References

- Eisenbud, David (1995), *Commutative algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Berlin, New York: Springer-Verlag, ISBN 3-540-94268-8, MR 1322960 (//www.ams.org/mathscinet-getitem?mr=1322960), Zbl 0819.13001 (http://www.zentralblatt-math.org/zmath/en/search /?format=complete&q=an:0819.13001)
- Hazewinkel, Michiel, ed. (2001), "Noether theorem" (http://www.encyclopediaofmath.org/index.php?title=n /n066790), *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4. NB the lemma is in the updating comments.
- Noether, Emmy (1926), "Der Endlichkeitsatz der Invarianten endlicher linearer Gruppen der Charakteristik p" (http://gdz.sub.uni-goettingen.de/no_cache/dms/load /img/?IDDOC=63971), *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen*: 28–35