Course Title:	Math 746
	Commutative Algebra
Semester:	**Fall 2024**
Time:	3:55 pm – 5:10 pm, Monday and Wednesday
Instructor:	Kustin
Textbook:	No textbook is required
References:	Commutative algebra with a view toward Algebraic Geometry, by Eisenbud,
	Introduction to Commutative Algebra by Atiyah and Macdonald,
	Commutative Algebra by Matsumura,
	Commutative Ring Theory by Matsumura,
Prerequisite	Math 702 (or consent of the graduate director)
Grades:	will be based on homework.

Commutative algebra is the branch of abstract algebra that studies commutative rings, their ideals, and modules over such rings. Both algebraic geometry and algebraic number theory build on commutative algebra. Prominent examples of commutative rings include polynomial rings, rings of algebraic integers, including the ordinary integers $\mathbb{Z}$, and p-adic integers.

Commutative algebra is the main technical tool in the local study of schemes.

Two main themes will be studied in Math 746: dimension and depth. The **dimension** of a commutative ring is an algebraic phenomenon (What is the length of the longest chain of prime ideals in the ring?), a geometric phenomenon (Is the ring the coordinate ring of a finite set of points?, a curve?, a surface?, a three-fold?, etc.), and a combinatorial phenomenon (Let R be a ring with unique maximal ideal $\mathfrak{m}$. For each non-negative integer n, the Hilbert function of R, evaluated at n, is $H(n)$ which is equal to the vector space dimension of $\mathfrak{m}^n/\mathfrak{m}^{n+1}$. For large n, the Hilbert function is a polynomial. What is the degree of this polynomial?) Much of the course will be spent defining these words carefully and proving the resulting theorem about dimension.

The other main concept in Math 746 is depth. The **depth** of the ring R with maximal ideal $\mathfrak{m}$ is the length of the longest regular sequence in $\mathfrak{m}$ on R. Regular sequences and depth play an important role because many properties pass across the ring homomorphism from R to R mod the regular sequence.

A ring with depth equal to dimension is called a Cohen-Macaulay ring. For such a ring one is able to mod out by a regular sequence and obtain a zero-dimensional ring. One often proves theorems about Cohen-Macaulay rings by induction on dimension. Many rings that arise in algebraic geometry are Cohen-Macaulay.

Consider the example $R = k[x,y,z,w]/(xy - zw)$, where k is an infinite field. The ring R has dimension 3. One of the saturated chains of prime ideals in R is

$$P_0 = (0) \subseteq P_1 = (x,w)R \subseteq P_2 = (x,y,w)R \subseteq P_3 = (x,y,z,w)R.$$

The ring R is the coordinate ring of the 3-fold X, where X is the set of all points (a_1, a_2, a_3, a_4) in k^4 which satisfy $a_1a_2 = a_3a_4$. The 3-fold X is a three dimensional geometric object. (One might also call it a hypersurface in 4-space. This hypersurface is "cut out by" one irreducible polynomial.) The Hilbert function of R is $H(n) = (n+1)^2$; this particular Hilbert function is a polynomial function for all n. (The Hilbert polynomial, as described above, always has degree one less than the dimension of the ring.) The ring R is Cohen-Macaulay. One regular sequence on R is $x, y, z+w$. The ring $R/(x, y, z+w)$ is equal to $k[x, y, z, w]/(x, y, z+w, z^2)$, which is an example of a zero dimensional ring. (I made the above calculation of the Hilbert function of R by hand using, essentially, Hilbert's original proof. But one can make such calculations on the computer using the Computer Algebra System Macaulay2. One can use Macaulay2 to experiment and make hard calculations long before one knows all of the theorems.)

It is our intention to offer a continuation of Math 746 in the Spring of 2025.

See the website

`https://people.math.sc.edu/kustin/teaching/746/746.html`

for more information.