Homework for Math 747

These problems are due on the last day of classes; however I encourage you to hand them to me as you do them. I might assign more problems at a later date.

1. Let R be an arbitrary (commutative noetherian) ring, let M and N be R-modules, and let **F** and **G** both be resolutions of M by projective R-modules. If

$$\mathbf{A}: \ \dots \to A_i \xrightarrow{a_i} A_{i-1} \to \dots$$

is a complex, let $H_i(\mathbf{A})$ denote the i^{th} homology of **A**, namely $\mathrm{Ker}(a_i)/\mathrm{Im}(a_{i+1})$. In class I told you that

$$(*) \qquad H_i(\mathbf{F} \otimes_R N) \cong H_i(\mathbf{G} \otimes_R N),$$

for all $i \geqslant 0$. I called the common value $\mathrm{Tor}^R_i(M,N)$. In this exercise I want you to prove (*).

(a) Let α: **A** → **B** be any map of complexes of R-modules. (In other words, there is a commutative diagram

$$\begin{array}{ccccccc} \dots \to & A_i & \xrightarrow{a_i} & A_{i-1} & \to \dots \\ & \downarrow{\alpha_i} & & \downarrow{\alpha_{i-1}} & \\ \dots \to & B_i & \xrightarrow{b_i} & B_{i-1} & \to \dots \end{array} \quad .)$$

Prove that there is an induced map $\overline{\alpha}_i$: $H_i(\mathbf{A}) \to H_i(\mathbf{B})$ for all i.

(b) Let α: **A** → **B** and β: **A** → **B** both be maps of complexes of R-modules. Suppose that α and β are homotopic. (In other words, suppose that for each subscript i, there is a map $s_i: A_i \to B_{i+1}$ such that $\alpha_i - \beta_i = s_{i-1}a_i + b_{i+1}s_i$ for all i.) Prove that α and β induce <u>the same</u> map on homology.

(c) Return to the Comparison Theorem which I proved in class. Let me remind you what I proved: Let $\dots \to A_1 \to A_0 \to M \to 0$, and $\dots \to B_1 \to B_0 \to N \to 0$ be complexes; if each A_i is projective, the second complex is exact, and there is a map $f\colon M \to N$, then there is a complex map α from $\mathbf{A}$ to $\mathbf{B}$ which covers f. Prove that if $\beta\colon \mathbf{A} \to \mathbf{B}$ also covers f, then α and β are homotopic.

(d) Prove (*). Hint: Use the comparison theorem to get a map $\alpha\colon \mathbf{F} \to \mathbf{G}$ which covers $id\colon M \to M$. Also get $\beta\colon \mathbf{G} \to \mathbf{F}$ covering $id\colon M \to M$. So $\beta\alpha$ and id are both maps from $\mathbf{F}$ to $\mathbf{F}$ which cover $id\colon M \to M$.

(e) As a consequence of (a) - (c) you can easily prove: If $f\colon M \to M'$ is an R-module map, then there is a map $f_*\colon \mathrm{Tor}^R_i(M,N) \to \mathrm{Tor}^R_i(M',N)$. Tell me how f_* is defined and why it is independent of various choices.

2. Fix a commutative ring R.

(a) Prove the Snake Lemma. Let:

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to 0 \\ & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & \\ 0 \to & & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \end{array}$$

be a commutative diagram of R-modules. Assume that the rows are exact. Prove that there is a long exact sequence:

$$\ker\alpha \to \ker\beta \to \ker\gamma \to \operatorname{cok}\alpha \to \operatorname{cok}\beta \to \operatorname{cok}\gamma.$$

Define the maps. Prove that they are well defined. Prove that the sequence is exact. (Needless to say, the interesting map is the one from $\ker\gamma$ to $\operatorname{cok}\alpha$. This map is called the connecting homomorphism.) Moreover, prove that if f is one-to-one, then $\ker\alpha \to \ker\beta$ is also. Finally, prove that if g' is onto, then $\operatorname{cok}\beta \to \operatorname{cok}\gamma$ is also.

(b) Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequences of complexes of R-modules. Prove that there is a long exact sequence of homology:

$$\cdots \to H_{i+1}(A'') \to H_i(A') \to H_i(A) \to H_i(A'') \to H_{i-1}(A') \to \cdots .$$

(Once again the interesting part is the connecting homomorphism.)

(c) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Prove that there is a long exact sequence of homology:

$$\cdots \to \mathrm{Tor}_{i+1}(M'',N) \to \mathrm{Tor}_i(M',N) \to \mathrm{Tor}_i(M,N) \to \mathrm{Tor}_i(M'',N) \to \mathrm{Tor}_{i-1}(M',N) \to \cdots .$$

3. Fix a commutative ring R. In class I told you that $\mathrm{Tor}^R(M,N)$ can be computed two different ways: One can resolve M, apply ___ $\otimes$ N, then take homology; or one can resolve N, apply M $\otimes$ ___, then take homology. In this exercise you will prove that both ways give the same answer. Let Tor(M,N) (with a capital T) represent the first method, and tor(M,N) (with a lower case t) represent the second method.

(a) Prove that $\mathrm{Tor}_i(M,N) = 0$ for all $i \geq 1$ if either of the modules M or N is a projective module. (The same argument shows that $\mathrm{tor}_i(M,N) = 0$ for all $i \geq 1$ if either of the modules M or N is a projective module. Don't bother to re-write your argument for this result.)

(b) Let **F** be a fixed projective resolution of M, and let **G** be a fixed projective resolution of N. Let K_0 be the kernel of $F_0 \to M$ and L_0 be the kernel of $G_0 \to N$. So we have short exact sequences:

$$0 \to K_0 \to F_0 \to M \to 0, \text{ and}$$
$$0 \to L_0 \to G_0 \to N \to 0 \quad .$$

Explain why the following commutative diagram has exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathrm{tor}_1(K_0,N) & & 0 & & \mathrm{tor}_1(M,N) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to \mathrm{Tor}_1(M,L_0) & \to & K_0 \otimes L_0 & \to & F_0 \otimes L_0 & \to & M \otimes L_0 & \to & 0 \\
 & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\
0 & \to & K_0 \otimes G_0 & \to & F_0 \otimes G_0 & \to & M \otimes G_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to \mathrm{Tor}_1(M,N) & \to & K_0 \otimes N & \to & F_0 \otimes N & \to & M \otimes N & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}
$$

Be sure to use Problems 2(c) (and the identical result for tor(M,___)) and 3(a).

(c) Apply the Snake Lemma to α, β, γ in order to conclude that $\mathrm{Tor}_1(M,N) \simeq \mathrm{tor}_1(M,N)$. (So $\mathrm{Tor}_1 = \mathrm{tor}_1$.)

(d) Do a quick diagram chase in order to prove that $\mathrm{tor}_1(K_0,N) \simeq \mathrm{Tor}_1(M,L_0)$.

(e) Now apply (c) and (d) to the short exact sequences:

$$0 \to K_i \to F_i \to K_{i-1} \to 0, \text{ and}$$

$$0 \to L_j \to G_j \to L_{j-i} \to 0.$$

in order to conclude $\mathrm{Tor}_1(K_i,L_{j-1}) \simeq \mathrm{Tor}_1(K_{i-1},L_j)$ for all i and j.

(f) Prove that $\mathrm{tor}_i(M,N) \simeq \mathrm{Tor}_i(M,N)$ for all i. (Hint: Why is $\mathrm{Tor}_n(M,N) \simeq \mathrm{Tor}_1(K_{n-1},N)$?)