The branches of a curve singularity

Thm (Zariski) Let T be the integral closure of the local one-dimensional domain (R, m). Suppose that T is f.g. as an R-module. Let $\hat{R}$ + $\hat{T}$ represent the completion of R and T in the m-adic topology. Then ~~$\hat{R}$ is reduced and~~ there is a 1-1 correspondence between the minimal prime ideals of $\hat{R}$ and the maximal ideals of T. (14a)

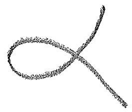
If M is a max ideal of T, then the corresp. min. prime of $\hat{R}$ is $\mathrm{Ker}\,(\hat{R} \longrightarrow \hat{T}_{M\hat{T}})$

[The argument I give comes (in some sense) from Kunz Introduction to Algebraic Curves Thm. 16.14.]

To exhibit examples it is help to know

$$\hat{T}_{M\hat{T}} = \widehat{T_M} \quad \uparrow \; M\text{-adic top}$$

Ex $y^2=x^2+x^3$



$\boxed{\text{The curve has 2 branches at } (0,0)}$ ✱

There are two geometric ways to observe ✱.
Each g. way may be translated into alg.
I plan to connect the 2 algebraic statements.

The first geometric app to branches

The ~~equation~~ poly $y^2-(x^2+x^3)$ is irred
but an analyst would say "so why think about polys, consider convergent power series instead"
holomorphic functions i.e.

and $y^2-(x^2+x^3) = y^2-x^2(1+x) = (y-xf)(y+xf)$

where f is a P.S. which squares to $1+x$

$$f=\sqrt{1+x}=1+\frac{1}{2}x+\frac{\frac{1}{2}(-\frac{1}{2})x^2}{2}+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})x^3}{3!}+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}x^4+\cdots$$

$$f^2=1+x+x^2\left(-\frac{1}{4}+\frac{1}{4}\right)+x^3\left(\frac{1}{8}-\frac{1}{8}\right)+x^4($$

$y-xf$ gives one branch of the curve at $(0,0)$
$y+xf$ gives the other branch

Algebraic translation Start with $R=\left(\frac{k[x,y]}{(y^2-x^2-x^3)}\right)_{(x,y)}$.

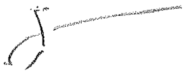
$\hat{R}=\frac{k[[x,y]]}{(y^2-x^2-x^3)}$. R is a domain, but $\hat{R}$ is not a domain,
but each minimal prime of $\hat{R}$ (i.e. $(y-xf),(y+xf)$) is a branch of $\hat{R}$

The second geometric app. to branches

⑯

Blow-up α until smooth

$k = \mathbb{C}$ (or some other alg. cl. field char $\neq 2$)



smooth

$$\mathcal{C} \xleftarrow{\pi} \mathcal{C}'$$

The branches of $\mathcal{C}$ at $(0,0)$ correspond to the points of $\pi^{-1}(0,0)$

The algebraic translation

Start with R (as before)

Find the integral closure $\bar{R}$ of R.

The max ideals of $\bar{R}$ represent the branches of $\mathcal{C}$ at P.

Note A 1-dim'l domain is regular $\Leftrightarrow$ normal.

For us $\frac{y}{x}$ is in $\mathbb{Q}f\,R$ and is integral over R

because $\left(\frac{y}{x}\right)^2 - 1 - x = 0$

(Note: $\frac{y}{x}$ is also the new variable after blowing up!)

We consider

$$\frac{k[x,y]}{(y^2-x^2-x^3)} \hookrightarrow \frac{k[x,y,\frac{x}{y}]}{(y^2-x^2-x^3)} \quad \text{(bir.) integ. extension.}$$

Now localize at $S = k[x,y] \setminus (x,y)$

$$R = S^{-1}\left(\frac{k[x,y]}{y^2-x^2-x^3}\right) \hookrightarrow S^{-1}\left(\frac{k[x,y,\frac{x}{y}]}{(y^2-x^2-x^3)}\right) \quad \text{is (bir.) integ. ext.}$$

I claim the ring on the right has 2 max ideals and

(Ring on right) = regular

$\therefore$ Rinjon visit $= \bar{R}$ and

\# of max ideals of $\bar{R}$ = # of min primes of $\hat{R}$

Cohen Seidenberg says that max of RHS $\cap$ R $= (x, y)$

If m is max of RHS then

$(x, y) \subseteq m \subseteq$

$x, \left(\left(\frac{y}{x}\right)^2 - 1 - x\right) \subseteq \mathfrak{m}$

so $\left(\frac{y}{x}\right)^2 - 1 \in \mathfrak{m}$

$\left(\frac{y}{x} - 1\right)^{0x}$ or $\left(\frac{y}{x} + 1\right)^{1x} \in \mathfrak{m}$

If $(R, \mathfrak{m})$ is local 1-dim domain, then $\exists$ 1-1 corresp between the min primes of $\hat{R}$ and the max ideals of $\bar{R}$.

~~And this correspondence is given by~~

~~$\hat{R} \longrightarrow \bar{R} \longrightarrow \hat{\bar{R}} \longrightarrow \hat{R}_{\mathfrak{m}}$~~

If $\mathfrak{m}$ is a max ideal of $\bar{R}$, then $\mathfrak{m}\hat{\bar{R}}$ is a max ideal of $\hat{\bar{R}}$ and the minimal prime of $\hat{R}$ which corresponds to $\mathfrak{m}$ is

$\operatorname{Ker}\left(\hat{R} \longrightarrow \hat{\bar{R}} \longrightarrow \left(\hat{\bar{R}}\right)_{\mathfrak{m}\hat{\bar{R}}}\right)$

~~where $R \subset \bar{R}$ is completed in $\mathfrak{m}_R$-adic top~~

where one completes $R \hookrightarrow \bar{R}$ in the $\mathfrak{m}_R$-adic topology so RHS has two max ideals and

$(RHS)_{\mathfrak{m}}$ is regular.

∴ RHS = T.

Now I want to show that

$\operatorname{Ker}\left(\hat{R} \longrightarrow \widehat{T_{\mathfrak{m}}}\right)$ picks up the min. primes of $\hat{R}$ as $\mathfrak{m}$ varies over the 2 max ideals of T.

$$(T)_{(x,\frac{y}{x}-1)} = \left(\frac{k[x,\frac{y}{x}]}{\left(\left(\frac{y}{x}\right)^2-1-x\right)}\right)_{x,\frac{y}{x}-1} = k\left[\tfrac{y}{x}-1\right]_{(\frac{y}{x}-1)}$$

$$x = \underbrace{\left(\tfrac{y}{x}-1\right)}_{\theta}(\theta+2)$$

$$\widehat{T_{(x,\frac{y}{y}-1)}} = k[[\theta]]$$

$$\hat{R} \longrightarrow \widehat{T_{(x,\frac{y}{x}-1)}}$$

$$x \longmapsto \theta^2+2\theta$$

$$y \longmapsto (\theta^2+2\theta)(\theta+1)$$

$$f \longrightarrow 1+\frac{\theta^2+2\theta}{2}-\frac{(\theta^2+2\theta)^2}{8}+\frac{(\theta^2+2\theta)^3}{16}-\frac{5(\theta^2+2\theta)^4}{128}+\cdots$$

$$=1+\theta+\theta^2\left(\tfrac{1}{2}-\tfrac{1}{2}\right)+\theta^3\left(-\tfrac{1}{2}+\tfrac{8}{16}\right)+\cdots$$

$$=1+\theta$$

So

$$y - xf \longmapsto (\theta^2+2\theta)(\theta+1) - (\theta^2+2\theta)(1+\theta) = 0$$

$$\hat{R} \longrightarrow \widehat{T_{(x,\frac{y}{x}+1)}}$$

$$\hat{T}_{(x,\frac{y}{x}-1)} = \left(\frac{k[x, \frac{y}{x}]}{(\frac{y}{x})^2-1-x}\right)_{(x,\frac{y}{x}+1)}$$

$$x \longmapsto (\varphi-2)\varphi$$

$$y \longmapsto (\varphi-1)(\varphi-2)\varphi$$

$$f \longmapsto 1+\frac{(\varphi-2)\varphi}{2} - \frac{(\varphi(\varphi-2))^2}{8} + \frac{(\varphi(\varphi-2))^3}{16} - \frac{5(\varphi(\varphi-2))^4}{128} + \cdots$$

$$= 1 - \varphi + \varphi^2\left(\frac{1}{2}-\frac{4}{8}\right) + \varphi^3\left(\frac{4}{8}-\frac{8}{16}\right) - $$

$$= 1-\varphi$$

$$y+xf \mapsto (\varphi-1)(\varphi-2)\varphi + (\varphi-2)\varphi(1-\varphi)$$

$$= 0$$

$$x = \underbrace{\left(\frac{y}{x}-1\right)}\underbrace{\left(\frac{y}{x}+1\right)}_{\varphi}$$

$$= (\varphi-2) \quad \varphi$$

$$y = x\left(\frac{y}{x}+1-1\right)$$

$$= \varphi(\varphi-2)(\varphi-1)$$

<u>Thm</u> (Zariski) $(R, \mathfrak{m})$ local 1-dim'l domain

$$T = \overline{R}$$

Assume T is f.g. R-module.

Let $\hat{R}$ + $\hat{T}$ be m-adic completions

Then

① ∃ 1-1 corresp

min primes of $\hat{R}$ $\longleftrightarrow$ max ideals of T

② If M is a max ideal of T, then the corresponding

min prime of $\hat{R}$ is

$$\mathrm{Ker}\left(\hat{R} \longrightarrow (\hat{T})_{M\hat{T}}\right).$$

<u>Pf</u>

Pf of Thm

$R \to T$ mfe $\Rightarrow \hat{R} \to \hat{T}$ is mfe $\hat{}$ means complete in m_R-adic top.
(In fact $\hat{T} = T \otimes_R \hat{R}$)

R local $\Rightarrow T$ semi-local ~~(Cohen-S~~

R local $\Rightarrow \hat{R}$ local $\Rightarrow \hat{T}$ semi-local

3 Facts about Completions of semi-local rings

1)
$$\begin{array}{ccc} M\,\mathrm{Spec}\, T & & M\,\mathrm{Spec}\, \hat{T} \\ M & \longmapsto & M\hat{T} \\ \mathfrak{m}_{\cap T} & \longleftarrow & \mathfrak{M} \end{array}$$
is 1-1 corr.

2) $\hat{T}_{M\hat{T}} = \widehat{T_M}$ ← This $\wedge$ is M-adic top

↑ This hat is m_R-adic top

3) Let $\mathfrak{M}_1 \ldots \mathfrak{M}_n$ be the max ideals of $\hat{T}$, then The natural map

$\hat{T} \to \hat{T}_{\mathfrak{M}_1} \times \cdots \times \hat{T}_{\mathfrak{M}_n}$

is an isomorphism.

i.e. $\exists\ \varepsilon_1 \ldots \varepsilon_n$ mutually orthogonal idempotents in $\hat{T}$ with $1 = \varepsilon_1 + \cdots + \varepsilon_n$ and $\hat{T}\varepsilon_i = \hat{T}_{\mathfrak{M}_i}$

These are elementary results. They are in both Eisenbud + Matsumura

~~Cohen~~ The Cohen-Seidenberg Theorems apply to $\hat{R} \to \hat{T}$ mfe $\hat{R}$ local

- Every maximal ideal of $\hat{T}$ contracts to $\mathfrak{m}_{\hat{R}}$
- Lying Over occurs i.e. if p is prime of $\hat{R}$, then $\exists\ \mathfrak{P} \in \mathrm{Spec}\,\hat{T}$ s.t. $\mathfrak{P} \cap \hat{R} = p$

We Know Spec $\hat{T}$. Every prime $\hat{T}_{\mathfrak{M}_1} \times \cdots \times \hat{T}_{\mathfrak{M}_n}$ ~~looks~~ is

$\hat{T}_{\mathfrak{M}_1} \times \cdots \hat{T}_{\mathfrak{M}_{i-1}} \times \text{prime} \times \cdots \hat{T}_{\mathfrak{M}_n}$

At most one ε_i not in $\mathfrak{M}$

$\mathfrak{m}_i = (\varepsilon_1, \varepsilon_2 \ldots, \varepsilon_{i-1}, t\varepsilon_i, \ldots \varepsilon_n)$

The primes of $\hat{R}$ are

$\mathfrak{n}_1 \cap \hat{R}$ $\mathfrak{n}_2 \cap \hat{R} \dots$ $\mathfrak{n}_a \cap \hat{R}$ *

Still to come

a) The primes of * are distinct

b) $\mathfrak{n}_i \cap \hat{R} = \ker\left(\hat{R} \longrightarrow \hat{T}_{\mathfrak{m}_i}\right)$

(b) is easy The map $\hat{R} \longrightarrow \hat{T}_{\mathfrak{m}_i}$ comes in two pieces

$$\begin{tikzcd} \hat{R} \arrow[r] & \hat{T} \arrow[r] & \hat{T}_{\mathfrak{m}_i} \end{tikzcd}$$

$\uparrow$

I am 1-1 (because $\hat{T} = \hat{R} \otimes_R T$ and $\hat{R} \otimes_R -$ is ff.)

I will show $\ker\left(\hat{T} \to \hat{T}_{\mathfrak{m}_i}\right)$ is $\mathfrak{n}_i$

View $\hat{T} \simeq \hat{T}_{\mathfrak{m}_1} \times \dots \times \hat{T}_{\mathfrak{m}_a}$ $\hat{T}_{\varepsilon_i} \simeq \hat{T}_{\mathfrak{m}_i}$

i.e. $\exists\ \varepsilon_1 \dots \varepsilon_a \in \hat{T}$ $1 = \varepsilon_1 + \dots + \varepsilon_a$

so $\varepsilon_i \notin \mathfrak{m}_i \Rightarrow \varepsilon_1 \dots \widehat{\varepsilon_i} \dots \varepsilon_a \in \mathfrak{m}_i$

$\Rightarrow \mathfrak{m}_i = (\varepsilon_1 \dots t_i\varepsilon_i \dots \varepsilon_a)$ where $t_i\varepsilon_i \mapsto$ generator ~~of~~ max ideal of $\hat{T}_{\mathfrak{m}_i}$

$\hat{T}_{\mathfrak{m}_i}$ is compl DVR

and $\mathfrak{n}_i = (\varepsilon_1 \dots \widehat{\varepsilon_i} \dots \varepsilon_a)$

Clear that $\mathfrak{n}_i \subseteq \ker \hat{T} \to \hat{T}_{\mathfrak{m}_i}$

ε_i ~~is unit in~~ $\in \hat{T} \setminus \mathfrak{m}_i$

$\varepsilon_i \cdot \mathfrak{n}_i = 0$

$\frac{\hat{T}}{\mathfrak{n}_i} \simeq \hat{T}_{\mathfrak{m}_i}$ is complete 1-dim local domain

$\frac{\hat{T}}{\mathfrak{m}} \simeq \hat{T}_{\mathfrak{m}_i}$ is 1-d local d. so $\mathfrak{n}_i \subseteq \ker$ primes to sae

To finish arg. we show

① tqr $\hat T = qf\,\hat T_{m_1} \times \cdots \times qf\,\hat T_{m_2}$ where $m_1, \dots, m_2$ are the max ideals of $\hat T$

so tqr $\hat T$ has 2 max ideals

② ~~$\mathrm{tqr}\,\hat R = \frac{\hat R_{J_1}}{J_1 \hat R_{J_1}} \times \cdots \times \frac{\hat R_{J_4}}{J_4 \hat R_{J_4}}$~~ where $J_1, \dots, J_4$ are the min primes of $\hat R$

so tqr $\hat R$ has 4 max ideals

③ tqr $\hat R$ = tqr $\hat T$

For 1 $\hat T = \hat T_{m_1} \times \cdots \times \hat T_{m_2}$

✓

each one is complete DVR

$U = \{(z_1, \dots z_2) \mid z_i \neq 0\}$ clear

$U^{-1}\hat T = (\text{complement of } 0)^{-1}\,\hat T_{m_1} \times \cdots \times (\quad)^{-1}\,\hat T_{m_2}$

$\hat T \longrightarrow qf(\hat T_{m_1}) \times \cdots \quad qf(\hat T_{m_2})$

nzdvs. sent

each elt of U goes to unit

$\therefore U^{-1}\hat T \longrightarrow (\qquad\qquad)$ nzds sent

obviously onto

② Every elt in $\underline{\mathfrak{m}}$ is reg on R hence on $\hat{R}$

$\therefore\ \mathfrak{m}\hat{R} \notin \operatorname{Ass}\hat{R}$

$0 \to R \xrightarrow{x} R$

$\hat{R}$ is R-flat

$0 \to \hat{R} \xrightarrow{x} \hat{R}$

$\therefore \operatorname{Ass}\hat{R} = \{J_1, \dots, J_n\}$

Z.D. on $\hat{R} = \cup J_i$

$\therefore\ \hat{R} \setminus \cup J_i$ = N.Z.D. $\hat{R}$

tqr $\hat{R} = (\hat{R}\setminus\cup J_i)^{-1}\hat{R}$

↗

Finite spectrum.
Every prime ideal is maximal

~~For such ring ring = ring $\mathfrak{m}_1 \times \dots \times$~~

$$\text{tqr } \hat{R} = (\hat{R}\setminus\cup J_i)^{-1}\ \hat{R}$$ I have n max ideals

Lemma

$$\underset{\substack{\uparrow \\ \text{local} \\ \text{domain}}}{R} \underset{\text{m.f.e.}}{\subseteq} S \subseteq qf(R) \implies tqr\, \hat{R} = tqr\, \hat{S}$$

$\hat{\ }$ means completion in m_R-adic topol.

Pf ① hypoth $\Rightarrow$ $\exists\, d \in R \setminus \{0\}$ $\quad dS \subseteq R$

② hypoth $\Rightarrow$ $\hat{S} = \hat{R} \otimes_R S$ $\quad \therefore\ d\hat{S} \subseteq \hat{R}$

③ ~~[illegible]~~ Every non-zero elt of R is regular on R, S $\hat{R}, \hat{S}$
(because $\hat{R}$ is ff R-mod)

$\subseteq$ $\quad \dfrac{z}{w}$ $\quad z \leftarrow$ in $\hat{R} \subseteq \hat{S}$ $\quad w \nwarrow$ in $\hat{R}$ reg on $\hat{R}$

Claim w reg on $\hat{S}$. If $\theta \in \hat{S}$, $w\theta = 0$, then $d\theta \in \hat{R}$ & $wd\theta = 0$

w reg on $\hat{R} \Rightarrow d\theta = 0$ in $\hat{R}$ but d reg on $\hat{S} \Rightarrow \theta = 0$ in $\hat{S}$

$\therefore\ w$ reg on $\hat{S}$

$\therefore\ \dfrac{z}{w} \in tqr\, \hat{S}$

$\supseteq$ $\quad \dfrac{z}{w} \in tqr\, \hat{S}$ $\quad z$: in $\hat{S}$ $\quad w$: in $\hat{S}$ reg on $\hat{S}$

$$\underset{\substack{\uparrow \\ \text{in } tqr\, \hat{S}}}{\frac{z}{w}} = \frac{\overset{\text{in } \hat{R}}{dz}}{\underset{\text{in } \hat{R}}{dw}}$$

d, w reg on $\hat{S} \Rightarrow dw$ reg on $\hat{R}$