FROBENIUS POWERS OF COMPLETE INTERSECTIONS

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Introduction

Let R be a commutative noetherian local ring of characteristic $p > 0$, and let $\phi\colon R \to R$ be the Frobenius endomorphism, $\phi(a) = a^p$. Each iteration ϕ^r defines on R a new structure of R-module, denoted ${}^{\phi^r}\!R$, for which $a \cdot b = a^{p^r} b$.

In 1969 Kunz [7, (3.3)] discovered that if R is regular, then ${}^{\phi^r}\!R$ is flat for all $r \geq 0$ and, conversely, that if R is reduced and ${}^{\phi^r}\!R$ is flat for some $r \geq 1$, then R is regular. Regularity is equivalent to the finiteness of the projective dimension of the R-module $k = R/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of R, so Kunz's theorem connects the homological properties of k and those of ϕ. To summarize further results along these lines, we let $c(R)$ denote the least integer s such that $(\boldsymbol{y} : \mathfrak{m}) \nsubseteq \mathfrak{m}^s$ for some maximal R-regular sequence $\boldsymbol{y}$ (such an s exists by the Artin-Rees Lemma).

For a finitely generated R-module M the following conditions are equivalent.

(i) M has finite projective dimension.

(ii) $\operatorname{Tor}_n^R(M, {}^{\phi^r}\!R) = 0$ for all $n, r \geq 1$.

(iii) $\operatorname{Tor}_n^R(M, {}^{\phi^r}\!R) = 0$ for all $n \geq 1$ and infinitely many r.

(iv) $\operatorname{Tor}_n^R(M, {}^{\phi^r}\!R) = 0$ for $j \leq n \leq j + \operatorname{depth} R + 1$ where j, r are fixed integers satisfying $j \geq 1$ and $r > \log_p(c(R))$.

The implication (i) $\implies$ (ii) is a fundamental theorem of Peskine and Szpiro [9, (1.7)]. An early converse, (iii) $\implies$ (i), was given by Herzog [4, (3.1)]. Recently, Koh and Lee [6, (2.6)] proved (iv) $\implies$ (i) (but stated a weaker result).

The local ring R is a *complete intersection* if in some (equivalently, in each) Cohen presentation of its $\mathfrak{m}$-adic completion as a homomorphic image of a regular local ring, the defining ideal is generated by a regular sequence. When R has this property and the length $\ell_R(M)$ is finite, a sharpening of Herzog's theorem is proved (but is not stated explicitly) in [8, (2.4)]: If $n \geq 1$ and M has infinite projective dimension, then $0 < \lim_{r\to\infty} \big(\ell_R\big(\operatorname{Tor}_n^R(M, {}^{\phi^r}\!R)\big)/p^{r \dim R}\big) < \infty$.

Our main result links, qualitatively and quantitatively, the homology of Frobenius powers of a complete intersection and the homology of the residue field.

Theorem. *Let M be a module over a local complete intersection ring $(R, \mathfrak{m}, k)$.*

If $\operatorname{Tor}_j^R(M, {}^{\phi^r}\!R) = 0$ for some fixed $j, r \geq 1$ then $\operatorname{Tor}_n^R(M, {}^{\phi^r}\!R) = 0$ for all $n \geq j$; if, furthermore, M is finitely generated, then M has finite projective dimension.

If M has finite length and infinite projective dimension, then for each $r \geq 1$ both

$$\lim_{s\to\infty} \frac{\ell_R\big(\operatorname{Tor}_{2s}^R(M, {}^{\phi^r}\!R)\big)}{\ell_R\big(\operatorname{Tor}_{2s}^R(M, k)\big)} \quad \textit{and} \quad \lim_{s\to\infty} \frac{\ell_R\big(\operatorname{Tor}_{2s+1}^R(M, {}^{\phi^r}\!R)\big)}{\ell_R\big(\operatorname{Tor}_{2s+1}^R(M, k)\big)}$$

are rational numbers, and at least one of them is positive.

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It should be noted that none of the conclusions of the theorems requires that R be a complete intersection. While we do not know whether this hypothesis is necessary, it does play a major role in our proofs. We use it in Proposition 4 to prove that ${}^{\phi^r}\!R$ is rigid, by refining techniques from [3], [8]. We invoke it again to apply results from [1], [2]: in Proposition 8 to deduce finite projective dimension from rigidity, and in Proposition 9 to study asymptotic behavior of Tor's.

1. Rigidity

Throughout our discussion, different module structures on the same abelian group will be induced by various homomorphisms of commutative rings. We start by describing notation that will keep track of the module structure in use.

If $\alpha\colon A \to B$ is a homomorphism of commutative rings, then ${}^{\alpha}B$ denotes the A-B-bimodule B with A acting through α and B acting through id_B, that is, $a\cdot b' = \alpha(a)b'$ and $b \cdot b' = bb'$ for all $a \in A$, $b' \in {}^{\alpha}B$, $b \in B$. For each A-module M the tensor product $M \otimes_A {}^{\alpha}B$ is a B-module: $b\cdot(m \otimes b') = m \otimes (bb')$ for all $b \in B$, $m \in M$, $b' \in {}^{\alpha}B$. Using a projective resolution of M to compute Tor's, one endows $\mathrm{Tor}_n^A(M, {}^{\alpha}B)$, for each $n \geq 0$, with a B-module structure that is natural in M.

We fix a prime number p and an integer $r > 0$, and set $q = p^r$. We assume all rings to be commutative noetherian of characteristic p, and for any such ring A we use φ to denote the r'th iteration of the Frobenius endomorphism: $\varphi(a) = a^q$ for all $a \in A$. For $\boldsymbol{a} = a_1, \dots, a_l$ with $a_i \in A$, we set $\boldsymbol{a}^q = a_1^q, \dots, a_l^q$.

From now on R denotes a local ring with maximal ideal $\mathfrak{m}$, residue field $k = R/\mathfrak{m}$, and $\mathfrak{m}$-adic completion $\widehat{R}$; note that the canonical map $\iota\colon R \to \widehat{R}$ satisfies $\iota\varphi = \varphi\iota$.

Standard use of the flatness of ι yields the isomorphisms below.

Remark 1. For each R-module M and all $n \geq 0$ there are natural isomorphisms

$$\mathrm{Tor}_n^R(M, {}^{\varphi}R) \otimes_R {}^{\iota}\widehat{R} \cong \mathrm{Tor}_n^R(M, {}^{\iota\varphi}\widehat{R}) = \mathrm{Tor}_n^R(M, {}^{\varphi\iota}\widehat{R}) \cong \mathrm{Tor}_n^{\widehat{R}}(M \otimes_R \widehat{R}, {}^{\varphi}\widehat{R})\,.$$

Choose, by Cohen's Structure Theorem, a surjective homomorphism $Q \to \widehat{R}$ where Q is a ring of formal power series $k[[\boldsymbol{t}]]$ on indeterminates $\boldsymbol{t} = t_1, \dots, t_e$.

Remark 2. Let $-'$ denote the functor $(- \otimes_Q {}^{\varphi}Q)$ from the category of Q-modules into itself; it is exact by Kunz's theorem. On the category of $\widehat{R}$-modules the functors $(-' \otimes_Q \widehat{R})$ and $(- \otimes_R {}^{\varphi}\widehat{R})$ are isomorphic, by associativity of tensor products.

We further assume that $\mathrm{Ker}(Q \to \widehat{R})$ is generated by a Q-regular sequence $\boldsymbol{x} = x_1, \dots, x_c$. The subquotients of the $(\boldsymbol{x})$-adic filtration of the Q-algebra $S = Q/(\boldsymbol{x}^q)$ are free $\widehat{R}$-modules, we can refine it to a filtration

$$0 = S_{q^c} \subset S_{q^c-1} \subset \cdots \subset S_1 \subset S_0 = S$$

with subquotients isomorphic to $\widehat{R}$. It defines exact sequences

$$(1_i) \qquad 0 \longrightarrow S_{i+1} \xrightarrow{\tau_i} S_i \xrightarrow{\sigma_i} R \longrightarrow 0 \quad \text{for} \quad i = 0, \dots, q^c - 1\,.$$

For each Q-module N and for $i = 0, \dots, q^c - 1$, set $S_i(N) = \mathrm{Ker}(N' \otimes_Q \tau_i)$.

The idea for the proof of part (b) below comes from [3, (2.2)] and [8, (2.1)].

Lemma 3. *If $R = \widehat{R}$, then for each R-module M the following hold.*

(a) *$S_i(M)$ is a homomorphic image of $S_0(M)$ for $i = 1, \dots, q^c - 1$.*

(b) *$S_0(M) \cong \mathrm{Tor}_1^R(M, {}^{\varphi}R)$.*

Proof. Applying $\operatorname{Tor}^Q(-,{}^{\varphi}R)$ to each sequences (1_i) we obtain isomorphisms

$$S_i(M)\cong \operatorname{Coker}\big(\operatorname{Tor}_1^Q(M',\sigma_i)\big)\quad\text{for}\quad i=0,\dots,q^c-1\,. \tag{2_i}$$

(a) As $S_0=S$, for each exact sequence of S-modules (1_i) there exists a map $\pi_i\colon S_0\to S_i$ with $\sigma_0=\sigma_i\pi_i$. In view of (2_i), it yields a commutative diagram

$$\begin{tikzcd}
\operatorname{Tor}_1^Q(M',S_0) \arrow[r, "\operatorname{Tor}_1^Q(M'{,}\sigma_0)"] \arrow[d, "\operatorname{Tor}_1^Q(M'{,}\pi_i)"'] & \operatorname{Tor}_1^Q(M',R) \arrow[r] \arrow[d, equal] & S_0(M) \arrow[r] \arrow[d, "\varpi_i"] & 0\\
\operatorname{Tor}_1^Q(M',S_i) \arrow[r, "\operatorname{Tor}_1^Q(M'{,}\sigma_i)"] & \operatorname{Tor}_1^Q(M',R) \arrow[r] & S_i(M) \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact, and so the homomorphism ϖ_i is surjective.

(b) Choose an exact sequence $0\longrightarrow K\xrightarrow{\kappa}L\xrightarrow{\lambda}M\longrightarrow 0$ with a free R-module L, then apply $(-\otimes_Q{}^{\varphi}Q)$ to get an exact sequence of Q-modules

$$0\longrightarrow K'\xrightarrow{\kappa'}L'\xrightarrow{\lambda'}M'\longrightarrow 0\,. \tag{3}$$

Writing $L=G\otimes_Q R$ with a free Q-module G, we obtain a commutative diagram

$$\begin{tikzcd}
 & G\otimes_Q\operatorname{Tor}_1^Q(S,S) \arrow[r, "G\otimes_Q\operatorname{Tor}_1^Q(S{,}\sigma)"] \arrow[d, "\cong"'] & G\otimes_Q\operatorname{Tor}_1^Q(S,R) \arrow[d, "\cong"]\\
\operatorname{Tor}_1^Q(L,R)\otimes_Q{}^{\varphi}Q \arrow[r, "\cong"] \arrow[d, "\operatorname{Tor}_1^Q(\lambda{,}R)\otimes_Q{}^{\varphi}Q"'] & \operatorname{Tor}_1^Q(L',S) \arrow[r, "\operatorname{Tor}_1^Q(L'{,}\sigma)"] \arrow[d, "\operatorname{Tor}_1^Q(\lambda'{,}S)"] & \operatorname{Tor}_1^Q(L',R) \arrow[d, "\operatorname{Tor}_1^Q(\lambda'{,}R)"]\\
\operatorname{Tor}_1^Q(M,R)\otimes_Q{}^{\varphi}Q \arrow[r, "\cong"] & \operatorname{Tor}_1^Q(M',S) \arrow[r, "\operatorname{Tor}_1^Q(M'{,}\sigma)"] & \operatorname{Tor}_1^Q(M',R)
\end{tikzcd}$$

with isomorphisms due to the flatness of G and ${}^{\varphi}Q$ over Q, and the equality $R'=S$.

The Koszul complex $\mathrm{K}(\boldsymbol{x},Q)$ is a free resolution of R over Q. For each R-module N the differential of the complex $N\otimes_Q\mathrm{K}(\boldsymbol{x},Q)$ is trivial, so there is an isomorphism $\operatorname{Tor}_1^Q(-,R)\cong(-\otimes_R R^c)$ of functors on the category of R-modules. In particular, $\operatorname{Tor}_1^Q(\lambda,R)$ is surjective, hence so is $\operatorname{Tor}_1^Q(\lambda',S)$. Similarly, the Koszul complex $\mathrm{K}(\boldsymbol{x}^q,Q)$ resolves S over Q. The differential of $\mathrm{K}(\boldsymbol{x}^q,Q)\otimes_Q N$ is trivial for each S-module N, so there is an isomorphism $\operatorname{Tor}_1^Q(S,-)\cong(S^c\otimes_S -)$ of functors on the category of S-modules. Thus, $\operatorname{Tor}_1^Q(S,\sigma)$ is surjective, hence so is $\operatorname{Tor}_1^Q(L',\sigma)$.

Formula (2_0) and the preceding computations yield isomorphisms

$$S_0(M)\cong\operatorname{Coker}\big(\operatorname{Tor}_1^Q(M',\sigma)\big)\cong\operatorname{Coker}\big(\operatorname{Tor}_1^Q(\lambda',R)\big)\,.$$

The exact sequence (3) induces the top row of the commutative diagram

$$\begin{tikzcd}
\operatorname{Tor}_1^Q(L',R) \arrow[r, "\operatorname{Tor}_1^Q(\lambda'{,}R)"] & \operatorname{Tor}_1^Q(M',R) \arrow[r] & K'\otimes_Q R \arrow[r, "\lambda'\otimes_Q R"] \arrow[d, "\cong"'] & L'\otimes_Q R \arrow[d, "\cong"]\\
0 \arrow[r] & \operatorname{Tor}_1^R(M,{}^{\varphi}R) \arrow[r] & K\otimes_R{}^{\varphi}R \arrow[r, "\lambda\otimes_R{}^{\varphi}R"] & L\otimes_R{}^{\varphi}R
\end{tikzcd}$$

with isomorphisms from Remark 2. It gives isomorphisms that finish the proof:

$$\operatorname{Coker}\big(\operatorname{Tor}_1^Q(\lambda',R)\big)\cong\operatorname{Ker}(\lambda'\otimes_Q R)\cong\operatorname{Ker}(\lambda\otimes_R{}^{\varphi}R)\cong\operatorname{Tor}_1^R(M,{}^{\varphi}R)\,. \qquad\square$$

Proposition 4. *If R is a complete intersection and M is an R-module such that $\operatorname{Tor}_j^R(M, {}^{\varphi}R) = 0$ holds for some $j > 0$, then $\operatorname{Tor}_n^R(M, {}^{\varphi}R) = 0$ for all $n \geq j$.*

Proof. In view of Remark 1 and the faithful flatness of $\iota\colon R \to \widehat{R}$, we may assume that R is complete. Obvious inductive considerations show that it suffices to establish the vanishing of $\operatorname{Tor}_{j+1}^R(M, {}^{\varphi}R)$. Replacing M by a $(j-1)$st syzygy, and adjusting notation, we may change our hypothesis to read $\operatorname{Tor}_1^R(M, {}^{\varphi}R) = 0$. Thus, the proposition will be proved once we show that this implies $\operatorname{Tor}_2^R(M, {}^{\varphi}R) = 0$.

The exact sequences (1_i) and (3), and Remark 2, yield commutative diagrams

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
0 \arrow[r] & S_i(K) \arrow[r] & K' \otimes_Q S_{i+1} \arrow[r, "K' \otimes_Q \tau_i"] \arrow[d, "\kappa' \otimes_Q S_{i+1}"'] & K' \otimes_Q S_i \arrow[r] \arrow[d, "\kappa' \otimes_Q S_i"] & K \otimes_R {}^{\varphi}R \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & L' \otimes_Q S_{i+1} \arrow[r] \arrow[d] & L' \otimes_Q S_i \arrow[r] \arrow[d] & L \otimes_R {}^{\varphi}R \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & M' \otimes_Q S_{i+1} \arrow[r] \arrow[d] & M' \otimes_Q S_i \arrow[r] \arrow[d] & M \otimes_R {}^{\varphi}R \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

for $i = 0, \dots, q^c - 1$. The rows are exact by Lemma 3. The columns are exact due to right exactness of tensor products and, for the rightmost one, to our hypothesis.

By decreasing induction on i we prove the labeled maps are injective. If $i = q^c - 1$, then $S_{i+1} = 0$, so all modules in the left hand column are trivial, and our assertion is clear. If $0 \leq i < q^c - 1$, then $\kappa' \otimes_Q S_{i+1}$ is injective by the induction hypothesis. Applying the Snake Lemma to the two top rows we see that $\kappa' \otimes_Q S_i$ is injective, then applying it to the two columns on the left we conclude that $K' \otimes_Q \tau_i$ is injective.

The injectivity of $K' \otimes_Q \tau_0$ yields $S_0(K) = 0$. Lemma 3 shows that $\operatorname{Tor}_1^R(K, {}^{\varphi}R)$ vanishes. This module is isomorphic to $\operatorname{Tor}_2^R(M, {}^{\varphi}R)$, so we are done. $\square$

2. Complexity

Let M and N be finitely generated modules over a local complete intersection ring R. Following [1], we say that the pair (M, N) has *complexity* d, and write $\operatorname{cx}_R(M, N) = d$, if d is the least non-negative integer with the property

$$\ell_R(\operatorname{Ext}_R^n(M, N) \otimes_R k) \leq \beta n^{d-1}$$

for some $\beta \in \mathbb{R}$ and all $n \gg 0$. As noted in [1, (1.3)], a result of Gulliksen [5, (3.1)] implies $\operatorname{cx}_R(M, N) \leq \operatorname{codim} R$, where $\operatorname{codim} R = \ell_R(\mathfrak{m}/\mathfrak{m}^2) - \dim R$.

The number $\operatorname{cx}_R(M, k)$ is called the complexity of M and is denoted $\operatorname{cx}_R M$. It measures the polynomial rate of growth of a minimal free resolution of M. In particular, $\operatorname{cx}_R M = 0$ if and only if M has finite projective dimension.

Remark 5. Inequalities, proved in [1, (5.7)], link various complexities:

$$\operatorname{cx}_R M + \operatorname{cx}_R N - \operatorname{codim} R \leq \operatorname{cx}_R(M, N) \leq \min\{\operatorname{cx}_R M\,,\ \operatorname{cx}_R N\}. \tag{4}$$

If M is a Cohen-Macaulay R-module, then [1, (5.6.2), (5.6.10)] yield an equality

$$\operatorname{cx}_R M = \operatorname{cx}_R \left(\operatorname{Ext}_R^{\dim R - \dim M}(M, R) \right) . \tag{5}$$

Identifying $\widehat{R} \otimes_Q {}^{\varphi}Q$ and $S = Q/(\boldsymbol{x}^q)$, let $\sigma\colon S \to \widehat{R}$ denote the canonical surjection, and let ρ be the composition of local homomorphisms

$$R \xrightarrow{\iota} \widehat{R} = \widehat{R} \otimes_Q Q \xrightarrow{\widehat{R}\otimes_Q \varphi} \widehat{R} \otimes_Q {}^{\varphi}Q = S\,.$$

To each R-module M we associate the S-module $M'' = M \otimes_R {}^{\rho}S$.

Remark 6. The homomorphism ρ is flat and satisfies $\varphi\iota = \sigma\rho$.

For each R-module M and all $n \geq 0$ there are natural isomorphisms

$$\operatorname{Tor}_n^R(M, {}^{\varphi}R) \otimes_R {}^{\iota}\widehat{R} \cong \operatorname{Tor}_n^S(M'', {}^{\sigma}\widehat{R})\,.$$

Indeed, ι is flat, and $\widehat{R} \otimes_Q \varphi$ is flat because $\varphi\colon Q \to Q$ is, so their composition ρ is flat. The equality $\varphi\iota = \sigma\rho$ is easily verified. The desired isomorphisms follow.

Lemma 7. *The ring S is a complete intersection, with* $\operatorname{codim} S = \operatorname{cx}_S {}^{\sigma}\widehat{R}$.

For each finitely generated S-module L the following equalities hold

$$\operatorname{cx}_S(L, {}^{\sigma}\widehat{R}) = \operatorname{cx}_S L = \operatorname{cx}_S({}^{\sigma}\widehat{R}, L)\,. \tag{6}$$

If M is a finitely generated R-module, then

$$\operatorname{cx}_S M'' = \operatorname{cx}_R M\,. \tag{7}$$

Proof. The sequence $\boldsymbol{x}^q = x_1^q, \dots, x_c^q$ is Q-regular and contained in $(\boldsymbol{t})^2$, so $S = Q/(\boldsymbol{x}^q)$ is a complete intersection of codimension c. In view of the inclusion $(\boldsymbol{x}^q) \subseteq (\boldsymbol{t})(\boldsymbol{x})$, a result of Tate [10, Theorem 6] provides the first equality below:

$$\sum_{n=0}^{\infty} \ell_R\big(\operatorname{Ext}_S^n({}^{\sigma}\widehat{R}, k)\big)t^n = \frac{(1+t)^c}{(1-t^2)^c} = \frac{1}{(1-t)^c} = \sum_{n=0}^{\infty} \binom{n+c-1}{c-1} t^n\,.$$

Thus, $\operatorname{cx}_S {}^{\sigma}\widehat{R} = c = \operatorname{codim} S$. From this expression and Remark 4 we get the equalities in (6). Equality (7) comes from the isomorphisms $\operatorname{Ext}_S^n(M'', k) \cong \operatorname{Ext}_R^n(M, k)$ which hold for all n, due to the flatness of ρ and ι. □

Proposition 8. *If M is a finitely generated R-module and $\operatorname{Tor}_j^R(M, {}^{\varphi}R) = 0$ for some $j > 0$, then M has finite projective dimension.*

Proof. Remark 6 and Proposition 4 yield $\operatorname{Tor}_n^S(M'', {}^{\sigma}\widehat{R}) = 0$ for $n \geq j$. By [1, Theorem III] this means $\operatorname{Ext}_n^S(M'', {}^{\sigma}\widehat{R}) = 0$ for $n \gg 0$, so $\operatorname{cx}_S(M'', {}^{\sigma}\widehat{R}) = 0$. Formulas (6) and (7) now give $\operatorname{cx}_R M = 0$, so M has finite projective dimension. □

Proposition 9. *If M has finite length and infinite projective dimension, then for each $r \geq 1$ the limits below are rational numbers, and at least one of them is positive:*

$$\lim_{s\to\infty} \frac{\ell_R\big(\operatorname{Tor}_{2s}^R(M, {}^{\varphi}R)\big)}{\ell_R\big(\operatorname{Tor}_{2s}^R(M, k)\big)} \quad \text{and} \quad \lim_{s\to\infty} \frac{\ell_R\big(\operatorname{Tor}_{2s+1}^R(M, {}^{\varphi}R)\big)}{\ell_R\big(\operatorname{Tor}_{2s+1}^R(M, k)\big)}\,.$$

Proof. Completion preserves length and projective dimension, so we may assume $R = \widehat{R}$. Let $\mathfrak{n}$ denote the maximal ideal of the local ring S, and let E be an injective envelope of the S-module $S/\mathfrak{n} \cong k$. The functor $\operatorname{Hom}_S(-, E)$ of Matlis duality is exact, and its restriction to the category of S-modules of finite length is isomorphic to the functor $\operatorname{Ext}_S^{\dim S}(-,S)$. Setting $N=\operatorname{Ext}_S^{\dim S}(M'',S)$, we obtain for each $n\geq 0$ an isomorphism of R-modules of finite length

$$\operatorname{Hom}_S\big(\operatorname{Tor}_n^S(M'',{}^{\sigma}R),E\big)\cong\operatorname{Ext}_S^n\big({}^{\sigma}R,\operatorname{Hom}_S(M'',E)\big)\cong\operatorname{Ext}_S^n({}^{\sigma}R,N)\,.$$

Using Remark 6 and the fact that Matlis duality preserves length, we get

$$\ell_R\big(\operatorname{Tor}_n^R(M,{}^{\varphi}R)\big)=\ell_S\big(\operatorname{Tor}_n^S(M'',{}^{\sigma}R)\big)=\ell_S\big(\operatorname{Ext}_S^n({}^{\sigma}R,N)\big)\,.$$

Gulliksen [5, (3.1)] shows that $\mathcal{N}=\operatorname{Ext}_S^*({}^{\sigma}R,N)$ is a finitely generated graded module over a polynomial ring $S[\boldsymbol{\chi}]$ with indeterminates $\boldsymbol{\chi}=\chi_1,\dots,\chi_c$ of degree 2. With $\mathcal{N}^+$ (respectively, $\mathcal{N}^-$) denoting the submodule of $\mathcal{N}$ consisting of all elements whose components of odd (respectively, even) degree are equal to 0, we have a direct sum decomposition $\mathcal{N}=\mathcal{N}^+\oplus\mathcal{N}^-$ of graded $S[\boldsymbol{\chi}]$-modules.

By the Hilbert-Serre Theorem, there exist polynomials $h_{\pm}(t)\in\mathbb{Q}[t]$ such that

$$\ell_S\big(\operatorname{Ext}_S^n({}^{\sigma}R,N)\big)=\begin{cases}h_+(n) & \text{for all}\quad n=2s\gg 0\,;\\ h_-(n) & \text{for all}\quad n=2s+1\gg 0\,;\end{cases}$$
$$\max\{\deg h_+(t)\,,\,\deg h_-(t)\}=\dim_{S[\boldsymbol{\chi}]}\mathcal{N}\,.$$

The S-module N has finite length, so it is annihilated by $\mathfrak{n}^m$ for some $m\geq 1$, hence $(\mathfrak{n}S[\boldsymbol{\chi}])^m\mathcal{N}=\mathfrak{n}^m\mathcal{N}=0$. This implies the first one in a sequence of equalities

$$\dim_{S[\boldsymbol{\chi}]}\mathcal{N}=\dim_{S[\boldsymbol{\chi}]}(\mathcal{N}/\mathfrak{n}\mathcal{N})=\operatorname{cx}_S({}^{\sigma}R,N)=\operatorname{cx}_S N=\operatorname{cx}_S M''=\operatorname{cx}_R M$$

where the second comes from dimension theory, the rest from formulas (6), (5), (7).

On the other hand, by [2, (8.1)] there exist polynomials $b_{\pm}(t)\in\mathbb{Q}[t]$ such that

$$\ell_R\big(\operatorname{Tor}_n^R(M,k)\big)=\begin{cases}b_+(n) & \text{for all}\quad n=2s\gg 0\,;\\ b_-(n) & \text{for all}\quad n=2s+1\gg 0\,;\end{cases}$$
$$\deg b_+(t)=\deg b_-(t)=\operatorname{cx}_R M\,.$$

The formulas displayed above clearly imply the desired assertions. □

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