MATH 702 – SPRING 2024
LOW LYING FRUIT – MARCH 13, 2024

(1) Let R be a (commutative) domain, I be an ideal in R, K be the quotient field of R, and $\phi : I \to R$ be an R-module homomorphism. Prove that there exists a K-module homomorphism $\Phi : K \to K$ such that $\Phi|_I = \phi$.

(2) Suppose $\boldsymbol{k} \subset E$ and $E \subseteq K$ are both finite dimensional Galois extensions. Does $\boldsymbol{k} \subseteq K$ have to be a Galois extension? Prove or give a counter example.

(3) Let $\boldsymbol{k} \subset K$ be a Galois extension of fields with $\dim_{\boldsymbol{k}} K = p^2$, for some prime integer p. Suppose E is a field with $\boldsymbol{k} \subseteq E \subseteq K$. Prove that $\boldsymbol{k} \subseteq E$ is a Galois extension.

(4) Give an example of a fields $\boldsymbol{k} \subseteq E \subseteq K$ with $\boldsymbol{k} \subseteq K$ a Galois extension of dimension p^3 for some prime integer p, but $\boldsymbol{k} \subseteq E$ not a Galois extension.

(5) Let $\boldsymbol{k}$ be a field of characteristic not equal to 2, $\boldsymbol{k} \subseteq K$ be a field extension with $\dim_{\boldsymbol{k}} K = 2$, and u be an element of K which is not in $\boldsymbol{k}$. Then the following statements hold.
   (a) The field extension $\boldsymbol{k} \subseteq K$ is Galois and the Automorphism group $\operatorname{Aut}_{\boldsymbol{k}} K$ is cyclic of order two.
   (b) The minimal polynomial of u over $\boldsymbol{k}$ is
   $$x^2 - (u + \tau(u))x + u\tau(u)$$
   in $\boldsymbol{k}[x]$, where τ is the non-identity element of $\operatorname{Aut}_{\boldsymbol{k}} K$.
   (c) The field K is equal to $\boldsymbol{k}(\Delta)$, with $\Delta^2 \in \boldsymbol{k}$, for $\Delta = u - \tau(u)$.

(6) Let $\boldsymbol{k} \subseteq K$ be a finite dimensional Galois extension of fields with $\operatorname{Aut}_{\boldsymbol{k}} K$ a cyclic group. Let σ be a generator of $\operatorname{Aut}_{\boldsymbol{k}} K$. Suppose that $E_1 \subseteq E_2$ are fields with
$$\boldsymbol{k} \subseteq E_1 \subseteq E_2 \subseteq K$$
and $\dim_{E_1} E_2 = 2$. If $u \in E_2 \setminus E_1$, then the minimal polynomial of u over E_1 is $(x - u)(x - \sigma(u))$.