MATH 702 – SPRING 2024
HOMEWORK 4

All modules in problems 13 and 14 are R-modules and all module homomorphisms in problems 13 and 14 are R-module homorphisms. In problem 13, I am thinking of R as a commutative ring.

13. Let $\phi : M \to P$ be a surjective homomorphism of R-modules. Suppose that P is a direct summand of a free R-module. Prove that P is a direct summand of M.
14. Let R be the ring $\mathbb{Z}[\sqrt{-5}]$ and let I be the ideal $(2, 1+\sqrt{-5})$ of R. Prove that I is a summand of a free R-module.
15. Let V be a vector space of dimension 8 over the field $\boldsymbol{k}$ and let $T : V \to V$ be a linear transformation with $T^8 = 0$. Suppose that v_0 is an element of V with the property that $\{T^i(v_0) \mid 0 \le i \le 7\}$ is a basis for V. Give the Jordan Canonical Form of T^i for each i, with $1 \le i \le 7$. For each i indicate the basis you use as you construct the Jordan Canonical Form of T^i.

    For example when I construct the Jordan Canonical Form of T, I use the basis

    $$v_0, T(v_0), T^2(v_0), T^3(v_0), T^4(v_0), T^5(v_0), T^6(v_0), T^7(v_0)$$

    and my Jordan Canonical Form is

    $$J_8(0) = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \end{bmatrix}.$$

    You may use a different convention for writing Jordan Canonical Form if you want, but do be sure to tell me what basis you are using for each matrix.