MATH 702 – SPRING 2024
HOMEWORK 4

All modules in problems 13 and 14 are R-modules and all module homomorphisms in problems 13 and 14 are R-module homorphisms. In problem 13, I am thinking of R as a commutative ring.

13. **Let $\phi : M \to P$ be a surjective homomorphism of R-modules. Suppose that P is a direct summand of a free R-module. Prove that P is a direct summand of M.**

The hypothesis ensures that there is an index set I, a free R-module $F = \bigoplus_{i\in I} Re_i$, and R-module homomorphisms $\pi : F \to P$ and $i : P \to F$ such that $\pi \circ i = \mathrm{id}_P$. Consider the picture

$$\begin{array}{ccc} & & F \\ & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\phi} & P. \end{array}$$

The homomorphism ϕ is surjective. For each $i \in I$ select $m_i \in M$ with $\phi(m_i) = \pi(e_i)$. We define an R-module homomorphism $\Phi : F \to M$ by sending e_i to m_i, for all i. Notice that Φ is a legitimate R-module homomorphism and $\phi \circ \Phi = \pi$. We finish the proof by observing that $\Phi \circ i$ is an R-module homomorphism from P to M with

$$\phi \circ (\Phi \circ i) = (\phi \circ \Phi) \circ i = \pi \circ i = \mathrm{id}_P\,.$$

each e_i.

14. **Let R be the ring $\mathbb{Z}[\sqrt{-5}]$ and let I be the ideal $(2, 1 + \sqrt{-5})$ of R. Prove that I is a summand of a free R-module.**

We define R-module homomorphisms $\pi : R^2 \to I$ and $i : I \to R^2$ such that $\pi \circ i = \mathrm{id}_I$.

Define π by $\pi\left(\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\right) = 2r_1 + (1 + \sqrt{-5})r_2$. It is obvious that π is an R-module homomorphism. Let K be the quotient field of R. Define $i : I \to K$ by $i(x) = \begin{bmatrix} xz_1 \\ xz_2 \end{bmatrix}$, where z_1 and z_2 are carefully chosen elements of K with the following properties:

- $z_1 I \subseteq R$,
- $z_2 I \subseteq R$, and
- $2z_1 + (1 + \sqrt{-5})z_2 = 1$.

The first two items ensure that the image of i is contained in R^2 and hence $i : I \to R^2$ is an R-module homomorphism. The third item ensures that $\pi \circ i$ is the identity map on I because if $x \in I$, then

$$(\pi \circ i)(x) = \pi\left(\begin{bmatrix} xz_1 \\ xz_2 \end{bmatrix}\right) = x\Big(2z_1 + (1 + \sqrt{-5})z_2\Big) = x.$$

It turns out that $z_1 = -1$ and $z_2 = \frac{1-\sqrt{-5}}{2}$ work. Indeed, $z_1 I \subseteq R$, $z_2(2) = 1-\sqrt{-5} \in R$, and $z_2(1+\sqrt{-5}) = 3 \in R$, and $2z_1 + (1+\sqrt{-5})z_2 = 1$.

15. **Let V be a vector space of dimension 8 over the field k and let $T : V \to V$ be a linear transformation with $T^8 = 0$. Suppose that v_0 is an element of V with the property that $\{T^i(v_0) \mid 0 \le i \le 7\}$ is a basis for V. Give the Jordan Canonical Form of T^i for each i, with $1 \le i \le 7$. For each i indicate the basis you use as you construct the Jordan Canonical Form of T^i.**

    For example when I construct the Jordan Canonical Form of T, I use the basis

$$v_0, T(v_0), T^2(v_0), T^3(v_0), T^4(v_0), T^5(v_0), T^6(v_0), T^7(v_0)$$

and my Jordan Canonical Form is

$$J_8(0) = \begin{bmatrix} 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0 \end{bmatrix}.$$

You may use a different convention for writing Jordan Canonical Form if you want, but do be sure to tell me what basis you are using for each matrix.

Let $\operatorname{diag}(A_1, \dots, A_r)$ be a block diagonal matrix with $A_1, \dots, A_r$ down the main diagonal, and zeros everywhere else.

- For T^2, I use the basis:

$$v_0, T^2(v_0), T^4(v_0), T^6(v_0) | T(v_0), T^3(v_0), T^5(v_0), T^7(v_0)$$

  The JCF is $\operatorname{diag}(J_4(0), J_4(0))$.
- For T^3, I use the basis:

$$v_0, T^3(v_0), T^6(v_0) | T(v_0), T^4(v_0), T^7(v_0) | T^2(v_0), T^5(v_0)$$

  The JCF is $\operatorname{diag}(J_3(0), J_3(0), J_2(0))$.
- For T^4, I use the basis:

$$v_0, T^4(v_0) | T(v_0), T^5(v_0) | T^2(v_0), T^6(v_0) | T^3(v_0), T^7(v_0)$$

  The JCF is $\operatorname{diag}(J_2(0), J_2(0), J_2(0), J_2(0))$.
- For T^5, I use the basis:

$$v_0, T^5(v_0) | T(v_0), T^6(v_0) | T^2(v_0), T^7(v_0) | T^3(v_0) | T^4(v_0)$$

  The JCF is $\operatorname{diag}(J_2(0), J_2(0), J_2(0), J_1(0), J_1(0))$.

- For T^6, I use the basis:
$$v_0, T^6(v_0)|T(v_0), T^7(v_0)|T^2(v_0)|T^3(v_0)|T^4(v_0)|T^5(v_0)$$
The JCF is $\mathrm{diag}(J_2(0), J_2(0), J_1(0), J_1(0), J_1(0), J_1(0))$.
- For T^7, I use the basis:
$$v_0, T^7(v_0)|T(v_0)|T^2(v_0)|T^3(v_0)|T^4(v_0)|T^5(v_0)|T^6(v_0)$$
The JCF is $\mathrm{diag}(J_2(0), J_1(0), J_1(0), J_1(0), J_1(0), J_1(0))$.