MATH 702 – SPRING 2024
HOMEWORK 3

9. Let M and N be $n \times n$ matrices over $\mathbb{Q}$. Suppose M and N are similar over $\mathbb{C}$. Prove M and N are similar over $\mathbb{Q}$. (Recall that the matrices M and N are similar over the field $\boldsymbol{k}$ if there exists an invertible matrix A with entries in $\boldsymbol{k}$ such that $M = ANA^{-1}$. In other words, the matrices M and N are similar over the field $\boldsymbol{k}$ if the linear transformation $\boldsymbol{k}^n \to \boldsymbol{k}^n$, which is given by $v \mapsto Mv$, is represented by the matrix N after a change of basis for $\boldsymbol{k}^n$.)
10. [1] (This is not a computer problem. Do not use the computer in parts (10a) or (10c). After you have (10a) and (10c), as you find (10b) and (10d), I do not mind if you use a computer to multiply polynomials or to multiply matrices times column vectors. You may also use the computer to calculate a determinant. Do not use the computer for anything other than those three processes.) Let $T : \mathbb{R}^8 \to \mathbb{R}^8$ be the linear transformation which is given by multiplication by the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

    (a) What is the Jordan canonical form of T?
    (b) What is the rational canonical form of T?
    (c) Find a basis $\mathscr{B}$ for $\mathbb{R}^8$ so that the matrix of T with respect to $\mathscr{B}$ is your answer to (10a).
    (d) Find a basis $\mathscr{C}$ for $\mathbb{R}^8$ so that the matrix of T with respect to $\mathscr{C}$ is your answer to (10b).
11. Let $T : V \to V$ be a linear transformation of a finite dimensional vector space over the field F. Suppose that the minimal polynomial of T is equal to the characteristic polynomial of T. Prove that V is cyclic as a $\boldsymbol{k}[x]$-module, (where $xv = T(v)$ for all $v \in V$).
12. Let $\boldsymbol{k}$ be a field, f be a polynomial in the ring $\boldsymbol{k}[x]$, and a be an element of $\boldsymbol{k}$.
    (a) Prove that $f(a) = 0$ if and only if f is in the ideal $(x - a)$.
    (b) Prove that f has at most $\deg f$ roots in $\boldsymbol{k}$.

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[1] I gave this problem in a previous course. In 2024 it is not important to me that you follow the precise instructions I gave in 2003. Nonetheless, these instructions do give a clear indication of which calculations are very easy to do by hand and which calculations I find irritating to do by hand.

(c) Let G be a finite subgroup of $(\boldsymbol{k} \setminus \{0\}, \times)$. Describe the group structure of the Abelian group G. Prove your answer. [2]

[2] We did this before. At that point we had not proven all of the pieces. Now we have proven all of the pieces. The purpose of this problem is to get you to think through and write down the various steps.