MATH 702 – SPRING 2024
HOMEWORK 3

9. **Let M and N be $n \times n$ matrices over $\mathbb{Q}$. Suppose M and N are similar over $\mathbb{C}$. Prove M and N are similar over $\mathbb{Q}$. (Recall that the matrices M and N are similar over the field $\boldsymbol{k}$ if there exists an invertible matrix A with entries in $\boldsymbol{k}$ such that $M = ANA^{-1}$. In other words, the matrices M and N are similar over the field $\boldsymbol{k}$ if the linear transformation $\boldsymbol{k}^n \to \boldsymbol{k}^n$, which is given by $v \mapsto Mv$, is represented by the matrix N after a change of basis for $\boldsymbol{k}^n$.)**

   Let M' and N' be the rational canonical forms of M and N (respectively) over $\mathbb{Q}$. Of course, M and M' are similar over $\mathbb{Q}$ and N and N' are similar over $\mathbb{Q}$. The matrices M and N are similar over $\mathbb{C}$; consequently, M and N have the same rational canonical, P, form over $\mathbb{C}$. On the other hand, the rational canonical form of M over $\mathbb{Q}$ is exactly the same as the rational canonical form of M over $\mathbb{C}$; so, $M' = P$. The exact same reasoning shows that $N' = P$.

10. [1] **(This is not a computer problem. Do not use the computer in parts (10a) or (10c). After you have (10a) and (10c), as you find (10b) and (10d), I do not mind if you use a computer to multiply polynomials or to multiply matrices times column vectors. You may also use the computer to calculate a determinant. Do not use the computer for anything other than those three processes.) Let $T : \mathbb{R}^8 \to \mathbb{R}^8$ be the linear transformation which is given by multiplication by the matrix**

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

   (a) **What is the Jordan canonical form of T?**
   (b) **What is the rational canonical form of T?**
   (c) **Find a basis $\mathscr{B}$ for $\mathbb{R}^8$ so that the matrix of T with respect to $\mathscr{B}$ is your answer to (10a).**
   (d) **Find a basis $\mathscr{C}$ for $\mathbb{R}^8$ so that the matrix of T with respect to $\mathscr{C}$ is your answer to (10b).**

[1] I gave this problem in a previous course. In 2024 it is not important to me that you follow the precise instructions I gave in 2003. Nonetheless, these instructions do give a clear indication of which calculations are very easy to do by hand and which calculations I find irritating to do by hand.

Let $e_1, \dots, e_8$ be the standard basis for $\mathbb{R}^8$. (In other words, A is the matrix of T with respect to $e_1, \dots, e_8$.) Let $\mathscr{B}$ be named $v_1, \dots, v_8$; and $\mathscr{C}$ be named $w_1, \dots, w_8$. Let B the the matrix of T with respect to $\mathscr{B}$ and C be the matrix of T with respect to $\mathscr{C}$.

The characteristic polynomial of T is $(x-2)^6(x-3)^2$. I will put the Jordan blocks that correspond to the eigenvalue 2 before the Jordan blocks that correspond to the eigenvalue 3. The nullity[2] of

$$A - 3I = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is 1. So there is one Jordan block associated to the eigenvalue 3 and that Jordan block is $J_2(3) = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$. The eigenvector for $J_2(3)$ is $v_8 = e_4$. The vector v_7 must satisfy $T(v_7) = 3v_7 + v_8$; so, v_7 is a solution of $(A-3I)v_7 = e_4$. We already calculated $A - 3I$. It is obvious that $(A-3I)(e_8) = e_4$. We take $v_7 = e_8$. The nullity of

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is 3; so B has 3 Jordan blocks associated to the eigenvalue 2. Three linearly independent eigenvectors which belong to the eigenvalue 2 are e_1, e_2, e_3. The nullity of $(A-2I)^2$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is 4; so J has exactly one Jordan block of size 2 or more. Keep in mind that

$$\text{nullity}(J_e(2) - 2I)^2 - \text{nullity}(J_e(2) - 2I) = \begin{cases} 0 & \text{if } e = 1 \\ 1 & \text{if } e \le 2. \end{cases}$$

[2]If T is a linear transformation of finite dimensional vector spaces, then the kernel of T is also a finite dimensional vector space. The nullity of T is the dimension of the kernel of T.

There is only one partition of 6 with 3 pieces and exactly one piece of size 2 or more; namely, $6 = 4 + 1 + 1$. I am ready to answer (a):

$$B = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

I look for v_3 with $T(v_3) = 2v_3 + z$ for some non-zero element z of the eigenspace of A which belongs to the eigenvalue 2. In other words, z is an element of the subspace of $\mathbb{R}^8$ which is spanned by e_1, e_2, e_3. I want v_3 with

$$(A - 2I)v_3 = \begin{bmatrix} * \\ * \\ * \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

with some $*$ not zero. Look at $A - 2I$. I pick $v_3 = e_5$ (and $v_4 = e_1$.) To find v_2, I solve $(A-2I)v_2 = v_3$. So, I take $v_2 = e_6$. To find v_1, I solve $(A-2I)v_1 = v_2$. So, I take $v_2 = e_7$. I pick $v_5 = e_2$ and $v_6 = e_3$. My answer to (c) is:

$$v_1 = e_7,\ v_2 = e_6,\ v_3 = e_5, v_4 = e_1,\ v_5 = e_2,\ v_6 = e_3,\ v_7 = e_8,\ v_8 = e_4.$$

I check that $Ae_7 = 2e_7 + e_6$, $Ae_6 = 2e_6 + e_5$, $Ae_5 = 2e_5 + e_1$, $Ae_1 = 2e_1$, $Ae_2 = 2e_2$, $Ae_3 = 2e_3$, $Ae_8 = 3e_8 + e_4$, and $Ae_4 = 3e_4$.

The invariant factors for T are $g_1 = x - 2$, $g_2 = x - 2$, and

$$g_3 = (x-2)^4(x-3)^2 = x^6 - 14x^5 + 81x^4 - 248x^3 + 424x^2 - 384x + 144.$$

(A computer expanded g_3 for me.) The answer to (b) is

$$C = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -144 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 384 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -424 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 248 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -81 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 14 \end{bmatrix}$$

The basis vectors w_1 and w_2 are just v_5 and v_6 (in either order). To be concrete, we take $\boxed{w_1 = v_5 = e_2}$ and $\boxed{w_2 = v_6 = e_3}$. Let V_3 be the T-invariant subspace of V which is generated by v_1 and v_7. The minimal polynomial of $T|_{V_3}$ is obviously equal to g_3, since

$(x-2)^4$ (which is the minimal polynomial of $T|_{(v_1,T(v_1),\dots)}$) and $(x-3)^2$, (which is the minimal polynomial of $T|_{(v_7,T(v_7),\dots)}$) both divide g_3 and g_3 is the minimal polynomial of T. If need be, we can use the Chinese Remainder Theorem to find a T-cyclic generator of V_3. On the other hand, I am confident that v_1+v_7 is a T-cyclic generator of the T-cyclic subspace generated by V_3. I'll just take $w_3 = v_1+v_7 = e_7+e_8$, $w_4 = T(w_3)$, $w_5 = T(w_4)$, $w_6 = T(w_5)$, $w_7 = T(w_6)$, $w_8 = T(w_7)$:

$$w_3 = \begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\1\end{bmatrix}, w_4 = \begin{bmatrix}0\\0\\0\\1\\0\\1\\2\\3\end{bmatrix}, w_5 = \begin{bmatrix}0\\0\\0\\6\\1\\4\\4\\9\end{bmatrix}, w_6 = \begin{bmatrix}1\\0\\0\\27\\6\\12\\8\\27\end{bmatrix}, w_7 = \begin{bmatrix}8\\0\\0\\108\\24\\32\\16\\81\end{bmatrix}, w_8 = \begin{bmatrix}40\\0\\0\\405\\80\\80\\32\\243\end{bmatrix}.$$

(A computer calculated $w_i = AW_{i-1}$ for $4 \le i \le 8$ for me.) Maybe the easiest way to verify that $w_1,\dots,w_8$ are linearly independent is to have a computer calculate that the determinant of

$$\begin{bmatrix}0&0&0&0&0&1&8&40\\1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&1&6&27&108&405\\0&0&0&0&1&6&24&80\\0&0&0&1&4&12&32&80\\0&0&1&2&4&8&16&32\\0&0&1&3&9&27&81&243\end{bmatrix}$$

is equal to 1. You might also want to verify that

$$Aw_8 - 14w_8 + 81w_7 - 248w_6 + 424w_5 - 384w_4 + 144w_3 = 0.$$

11. **Let $T : V \to V$ be a linear transformation of a finite dimensional vector space over the field F. Suppose that the minimal polynomial of T is equal to the characteristic polynomial of T. Prove that V is cyclic as a $\boldsymbol{k}[x]$-module, (where $xv = T(v)$ for all $v \in V$).**

Think about the rational canonical form of T. There are monic polynomials $g_1,\dots,g_s$ in $F[x]$ and T-cyclic subspaces $V_1,\dots,V_s$ of V, with $g_1|\dots|g_s$, the minimal polynomial of T equal to g_s, the characteristic polynomial of T equal to $\prod_{i=1}^{s} g_i$, the minimal polynomial of $T|_{V_i} = g_i$ for all i, and $\bigoplus_{i=1}^{s} V_i = V$. The minimal polynomial of T is equal to the characteristic polynomial of T; therefore, $s=1$ and the T-cyclic subspace V_1 is equal to all of V.

12. **Let $\boldsymbol{k}$ be a field, f be a polynomial in the ring $\boldsymbol{k}[x]$, and a be an element of $\boldsymbol{k}$.**
   (a) **Prove that $f(a) = 0$ if and only if f is in the ideal $(x-a)$.**

(b) **Prove that f has at most $\deg f$ roots in $\boldsymbol{k}$.**
(c) **Let G be a finite subgroup of $(\boldsymbol{k} \setminus \{0\}, \times)$. Describe the group structure of the Abelian group G. Prove your answer.** [3]

(12a). Apply the division algorithm to the pair f and $x-a$ to obtain $g(x) \in \boldsymbol{k}[x]$ and $\alpha \in \boldsymbol{k}$ with

$$f(x) = (x-a)g(x) + \alpha.$$

Observe that $f(a) = \alpha$. It follows that $f(a) = 0$ if and only if $\alpha = 0$. Thus, $f(a) = 0$ if and only if f is in the ideal $(x-a)$.

(12b). Let $a_1, \dots, a_s$ be the distinct roots of $f(x)$. Apply (12a) to see that

$$f(x) = (x - a_1)f_1(x)$$

for some $f_1(x) \in \boldsymbol{k}[x]$. Observe that

$$0 = f(a_2) = (a_2 - a_1)f_1(a_2).$$

The element $a_2 - a_1$ of $\boldsymbol{k}$ is not zero; hence, $f_1(a_2) = 0$. Apply (12a) again to obtain $f_2(x) \in \boldsymbol{k}[x]$ with $f(x) = (x-a_1)(x-a_2)f_2(x)$. Repeat this procedure s times to obtain

$$f(x) = \prod_{i=1}^{s}(x - a_i)f_s(x)$$

for some $f_s(x) \in \boldsymbol{k}[x]$. Observe that

$$s \le s + \deg f_s = \deg f.$$

(12c). According to the structure theorem for finitely generated Abelian groups, there are elements $g_1, \dots, g_s$ in G so that the group G is equal to the direct product

$$\langle g_1 \rangle \times \cdots \times \langle g_s \rangle$$

and

$$\operatorname{ord}(g_1) | \operatorname{ord}(g_2) | \cdots | \operatorname{ord}(g_s),$$

where $\operatorname{ord}(g)$ is the order of the element g of G. It follows that

$$|G| = \prod_{i=1}^{s} \operatorname{ord}(g_i) \quad \text{and} \quad g^{\operatorname{ord}(g_s)} = 1,\ \forall g \in G.$$

Apply part (12b) to see that $|G| \le \operatorname{ord}(g_s)$; hence, $s = 1$ and G is cyclic.

[3] We did this before. At that point we had not proven all of the pieces. Now we have proven all of the pieces. The purpose of this problem is to get you to think through and write down the various steps.