MATH 702 – SPRING 2024
HOMEWORK 2
DUE WEDNESDAY, JANUARY 31, 2024 BY THE BEGINNING OF CLASS.

5. Prove that a finitely generated module over a (commutative) Noetherian ring is Noetherian. The ring R is Noetherian if the ideals of R satisfy the Ascending Chain Condition. The R-module M is Noetherian if the R-submodules of M satisfy the Ascending Chain Condition.
6. Let $\boldsymbol{k}$ be a field and $R = \boldsymbol{k}[x, y, z, w]$ and $S = \boldsymbol{k}[s_0, s_1, t_0, t_1]$ be polynomial rings over $\boldsymbol{k}$. Let $\phi : R \to S$ be the ring homomorphism with $\phi(x) = s_0t_0$, $\phi(y) = s_1t_1$, $\phi(z) = s_1t_0$, $\phi(w) = s_0t_1$, and the restriction of ϕ to $\boldsymbol{k}$ is the identity map. Prove that the kernel of ϕ is the ideal $(xy - zw)$ of R. The direction $(xy - zw) \subseteq \ker \phi$ is obvious. Your job is to prove the other inclusion. (This problem yields an algebraic proof that
$$\frac{\boldsymbol{k}[x, y, z, w]}{(xy - zw)}$$
is the homogeneous coordinate ring of the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$.)
7. Prove Eisenstein's criterion for irreducibility. Let $f = a_0 + \cdots + a_nx^n$ be a primitive polynomial in $\mathbb{Z}[x]$. Suppose there is a prime integer p with $p|a_i$ for $0 \leq i \leq n-1$, but $p^2 \nmid a_0$ and $p \nmid a_n$. Prove that f is an irreducible polynomial in $\mathbb{Q}[x]$.
8. Let p be a prime integer. Prove that the polynomial $f = x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible in $\mathbb{Q}[x]$. Hint: Observe that $f(x) = \frac{x^p-1}{x-1}$ and that $f(x)$ is irreducible if and only if $f(x+1)$ is irreducible.