MATH 702 – SPRING 2024
HOMEWORK 2
DUE WEDNESDAY, JANUARY 31, 2024 BY THE BEGINNING OF CLASS.

5. **Prove that a finitely generated module over a (commutative) Noetherian ring is Noetherian. The ring R is Noetherian if the ideals of R satisfy the Ascending Chain Condition. The R-module M is Noetherian if the R-submodules of M satisfy the Ascending Chain Condition.**

   It suffices to show that R^ℓ is Noetherian for each positive integer ℓ. (Indeed M is a quotient of R^ℓ for some ℓ. Every quotient of a Noetherian module is Noetherian. If $A \subseteq B$ are R-modules, then submodules of B/A all have the form C/A where C is a submodule of B which contains A. If B is a Noetherian module, then B/A is a Noetherian module.) Let N be a submodule of R^n. Consider the projection $\mathrm{proj} : R^\ell \to R$ which is given by

$$\mathrm{proj}\left(\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}\right) = r_1.$$

   Observe that $\mathrm{proj}(N)$ is an ideal of R; thus, $\mathrm{proj}(N)$ is finitely generated. It follows that there are elements $n_1, \dots, n_\#$ in N so that $N = R(n_1, \dots, n_\#) + N'$ where every element of N' has the form

$$\begin{bmatrix} 0 \\ * \\ \vdots \\ * \end{bmatrix}.$$

   It follows by induction on ℓ that N' is finitely generated. Therefore, N is also finitely generated.

6. **Let $\boldsymbol{k}$ be a field and $R = \boldsymbol{k}[x, y, z, w]$ and $S = \boldsymbol{k}[s_0, s_1, t_0, t_1]$ be polynomial rings over $\boldsymbol{k}$. Let $\phi : R \to S$ be the ring homomorphism with $\phi(x) = s_0t_0$, $\phi(y) = s_1t_1$, $\phi(z) = s_1t_0$, $\phi(w) = s_0t_1$, and the restriction of ϕ to $\boldsymbol{k}$ is the identity map. Prove that the kernel of ϕ is the ideal $(xy - zw)$ of R. The direction $(xy-zw) \subseteq \ker\phi$ is obvious. Your job is to prove the other inclusion. (This problem yields an algebraic proof that**

$$R = \frac{\boldsymbol{k}[x, y, z, w]}{(xy - zw)}$$

   is the homogeneous coordinate ring of the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$.)

Consider the following list of monomials in S:

$$\begin{array}{ll} (s_0t_0)^i(s_1t_1)^j & 0 \le i,j \\ (s_0t_0)^i(s_1t_1)^j(s_0t_1)^k & 0 \le i,j \text{ and } 1 \le k \\ (s_0t_0)^i(s_1t_1)^j(s_1t_0)^k & 0 \le i,j \text{ and } 1 \le k. \end{array}$$

Observe that no monomial on the above list is repeated! (Each monomial in the first line has the same degree in s_1 and t_1. The monomials in the second line have more degree in t_1 than they have in s_1. The monomials in the third line have more degree in s_1 than they have in t_1.) In particular, the monomials in this list are linearly independent elements of S over $\boldsymbol{k}$. If some element f of S is a $\boldsymbol{k}$ linear combination of monomials from the above list, then f is the zero element of S if and only if each coefficient of f is zero.

Here is the clever observation. Every element of R can be written as a polynomial which does not involve the product zw plus an element of $(xy - zw)$. In other words, every element of R is a linear combination of

$$\begin{array}{ll} x^iy^j & 0 \le i,j \\ x^iy^jw^k & 0 \le i,j \text{ and } 1 \le k \\ x^iy^jz^k & 0 \le i,j \text{ and } 1 \le k. \end{array}$$

plus an element of $(xy - zw)$. The homomorphism ϕ carries the listed monomials from R to the listed monomials from S. It follows that an element g of R is in the kernel of ϕ if and only if g is in the ideal $(xy - zw)$.

7. **Prove Eisenstein's criterion for irreducibility. Let $f = a_0 + \cdots + a_nx^n$ be a primitive polynomial in $\mathbb{Z}[x]$. Suppose there is a prime integer p with $p|a_i$ for $0 \le i \le n-1$, but $p^2 \not| a_0$ and $p \not| a_n$. Prove that f is an irreducible polynomial in $\mathbb{Q}[x]$.**

We proved in class (Cor. 3.44.(b)) that a primitive polynomial in $\mathbb{Z}[x]$ is irreducible in $\mathbb{Z}[x]$ if and only if it is irreducible in $\mathbb{Q}[x]$. Consequently, we need only show that f is irreducible in $\mathbb{Z}[x]$. Suppose that $f = gh$ in $\mathbb{Z}[x]$ with g and h primitive polynomials in $\mathbb{Z}[x]$. We must show that either g or h is a constant.

Write $g = \sum_i g_ix^i$ and $h = \sum h_ix^i$, with g_i and h_i in $\mathbb{Z}$. The hypothesis tells us that p divides $a_0 = g_0h_0$. Let us say that $p|g_0$. The hypothesis also tells us that $p^2 \not| a_0$; hence, $p \not| h_0$. The hypothesis also tells us that $p \not| a_n$; hence there is a least index r with $p \not| g_r$. We look at the coefficient

$$a_r = g_rh_0 + g_{r-1}h_1 + \dots \ .$$

We see that $p \not| a_r$; thus $r = n$ and $h = h_0$ is a constant.

8. **Let p be a prime integer. Prove that the polynomial $f = x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible in $\mathbb{Q}[x]$. Hint: Observe that $f(x) = \frac{x^p-1}{x-1}$ and that $f(x)$ is irreducible if and only if $f(x+1)$ is irreducible.**

I have made both observations. Let $g(x) = f(x+1)$. I show that $g(x)$ is irreducible. I see that

$$g(x) = \frac{(x+1)^p - 1}{(x+1) - 1} = \frac{\sum_{i=0}^{p} \binom{p}{i} x^i - 1}{x} = \frac{x^p + \binom{p}{p-1} x^{p-1} + \cdots + \binom{p}{2} x^2 + px}{x}$$
$$= x^{p-1} + \binom{p}{p-1} x^{p-2} + \cdots + \binom{p}{2} x + p.$$

Apply the Eisenstein criteria to see that g (and hence f) is irreducible.