ALGEBRA I AND II 2019–2020, A. KUSTIN, CLASS NOTES

1. THE RULES AND THE COURSE OUTLINE.

1.A. **Homework.** I will assign and grade homework. Please take it seriously. Please turn the homework in on-time.

Please type your solutions and e-mail me a .pdf version.

Each student should write up a solution to each problem (even if some problems were solved by a group of students.) Write in a professional manner. Make clear claims (not vague claims.) Write in complete sentences; use proper English; define all new terms carefully and completely; and define all notation carefully and completely.

It is a good idea to do some problems on your own. It is also valuable (and much fun) to work with other people. It is legal to look things up as you are doing homework; indeed, if you look hard enough, you can probably find the solution to anything I am likely to ask some place on the Internet.

Arrive at your solution however you like: by yourself, using the Internet, working with other people. Write the answer up by yourself in your own words. Make your answer as clear, understandable, and complete as possible. Give all details in your answer. Explain all notation and new concepts that you use. **Acknowledge all of your sources.**

1.B. **Exams.** There will be three in-class exams (Wednesday, September 20 Wednesday, October 18, and Monday, November 20) and one final exam (Friday, December 15, 4:00–6:30 PM). I'll ask some questions. You will do the best you can with them.

1.C. **Class Attendance.** I expect my students to attend class.

1.D. **Office Hours.** My office hours are 5:15–6:30 Monday and Wednesday.

I do respond to e-mail.

1.E. **References.** I will post my lecture notes on the web. These notes will serve as the text book for the course. I will re-write the notes as the course progresses. One of the main sources will be the notes I used in the past when I taught the course. These older notes were mainly taken from Jacobson [7]. (Dover has reprinted an inexpensive version of this book.) Jacobson was an excellent mathematician and he writes well. (He includes some topics that I don't like; I just skip them.) Also I taught the course from Artin [2] a few times. Artin is a very excellent mathematician. (Artin's book might be pegged at a slightly lower level than the course you are taking. The charming thing about it is that it covers topics that are not usually covered in the first year Algebra course.) In the 1990's every one took Algebra from Hungerford's book [6]; but I am not particularly fond of it. I took many courses from Rotman when I was a graduate student and a copy of an early edition of [9] sits on my desk. I know that other books by Rotman have lots of motivation and are well-organized. There is a decent chance that some of your course will come from [9]. Professors Ballard and Duncan taught 701 from Aluffi [1] in 2015–2016 and 2016–2017, respectively. It is a much different treatment of the material than the course I will teach. I do not have access to a copy. Professors Thorne and Vraciu taught from Dummit and Foote in 2017–2018 and 2018–2019, respectively. Dummit and Foote is the book "everybody learns Algebra from" now-a-days. I do have e-access to it. I may open it during your course; I may not.

The main thing is, make sure you know the name of the topic we are studying at any given time. Once you know the name the topic, then the Internet will lead you to all sorts of treatments of the topic. Find one treatment (from class, or from the Internet, or from some textbook) that resonates with you and then learn the topic very thoroughly.

If I happen to be following some source fairly closely, I will let you know (and if I forget, feel free to ask).

1.F. **What we study.** [1]

We study groups, rings, and fields. I think it is important to keep in mind that these notions are not handed down from on high; they grew organically.

A group is a set of invertible functions from a set to itself; this set is closed under composition. The set of all permutations of a finite set is the prototype of a group. Lagrange (1770) was one of the first to think about the set of permutations. Galois (1830) used groups of permutations as a way of describing which polynomials (in one variable with rational coefficients) can be "solved by radical". Felix Klein (1870) thought about "symmetry groups" of geometric objects. Groups were used by Gauss (1777-1855), Kronecker (1823-1891), and Kummer (1810-1893) in projects involving number theory. I visualize that Lagrange proved results about permutations, Klein proved results about symmetries of geometric objects, and Galois, Gauss, Kronecker, and Kummer proved results about number theory; before Cayley (1854) said "Hey! All of you proved the same result and it does not have anything to do with permutations, geometric objects, or number theory. It holds whenever one has … " At this point Cayley gave the abstract axioms for a group.

I think the idea of algebra is "Lets focus on the essential underlying idea rather than the specific example that we seem to be studying."

Commutative ring theory has a similar history. The main focus of number theory in the nineteenth history was to obtain a proof of Fermat's Last Theorem (that there do not exist positive integers a, b and c with $a^n + b^n = c^n$ when n is an integer at least three.) Fermat (1607-1665) wrote "I have discovered a truly remarkable proof of this theorem which this margin is too small to contain" in his copy of Arithmetica of Diophantus. One style of argument was to factor $x^n + y^n - z^n$ over $\mathbb{Z}$ with all of the n^{th} roots of one adjoined. This style of argument works when n is a prime integer and the "coefficient ring" is a "Unique Factorization Domain" (UFD). Alas, when $n = 23$, the coefficient ring is not a UFD. (Andrew Wiles proved Fermat's Last Theorem in 1995.)

[1] This material is mainly taken from [8, 10, 11].

In the meantime algebraic geometers were thinking about curves, surfaces, three-folds, etc. One way to study a geometric object X is to consider all of the (appropriate) functions from X back to the base field (say $\mathbb{R}$). Algebraic geometers especially care about polynomial maps. Differential Geometers and Functional Analysts probably want continuous maps. In all of these cases, the set of functionals (the set of maps from the geometric object to the base field) automatically form a ring. If f and g are functions from X to $\mathbb{R}$, then define $f + g : X \to \mathbb{R}$ to be the map that sends $x \in X$ to $f(x) + g(x)$ and define $f \times g$ from X to $\mathbb{R}$ to be $(f \times g)(x) = f(x) \cdot g(x)$. Algebraic Geometers call the set of functionals on X the coordinate ring of X; Functional Analysts call the set of functionals on X the dual Banach space of X.

Dedekind (or maybe Kronecker) observed you number theorists and you algebraic geometers are really proving the same theorems and the results are not really about number theory or curves, surfaces, and three-folds, they are really results about ... and at this point he defined an abstract ring.

The fact that fields were studied long before the official definition of field was given is quite clear. As humanity wanted to measure more quantities, do calculus, and solve more equations, humanity understood the (field of) rational numbers, the (field of) constructible numbers, the (field of) real numbers and (the field) of complex numbers. The official definition of an abstract field is probably due to Weber (1893) although Dedekind (1871) had an algebraic version and Kronecker (1881) had a more analytic version.

1.G. **Actions.** Groups act on sets. (In fact, so far I have only said that a group is a set of invertible functions from a set to itself.) One learns about the set by way of this group action and learns about the group by way of this group action. It is quite amazing.

Rings act on modules. A module is an Abelian group with a scalar multiplication. If R is the ring then the direct sum of copies of R (for example $R \oplus R$) is an R-module and any subset of an R-module which is closed under addition and scalar multiplication is another R-module.

A field is a special kind of ring. So, every field also acts on modules. It turns out that every module over a field $\boldsymbol{k}$ is a direct sum of copies of $\boldsymbol{k}$. Modules over a field are called vector spaces.

1.H. **Some of the highlights of the course.**

1. Groups
   - (a) We prove the Sylow Theorems about finite groups. Given a finite group we predict the sizes of some of its subgroups and we give information about how many such subgroups exist. These results are established by cleverly examining actions of the finite group.
   - (b) We study "solvable groups" and we prove that the group of all permutations of a five element set is not solvable. (We pick this idea up again in 1.1.(3d).)
2. Rings
   - (a) We study Principal Ideal Domains (PID). The ring of integers and the ring of polynomials over a field are examples of PIDs.
   - (b) We prove that every PID is a Unique Factorization Domain. (Keep in mind that the notion of unique factorization is central to both Number Theory and Algebraic Geometry.)

(c) We find the structure of all finitely generated modules over a PID. (Keep in mind that a finitely generated module over a field is just a finite dimensional vector space. One can write down a basis for such a thing. It is easy. A finitely generated module over an arbitrary ring might be very complicated. But there is structure theorem for a finitely generated module over a PID. This is an awesome theorem.) Here are two applications of this theorem.

(i) We record the structure of all finitely generated Abelian groups.

(ii) We record the canonical forms for matrices. Let $\boldsymbol{k}$ be the field of complex numbers, V be an n-dimensional vector space over $\boldsymbol{k}$, and $T : V \to V$ be a linear transformation. One would like a basis for V that makes T as pretty as possible. Maybe T is diagonalizable; that would be pretty. In general, the Jordan canonical form of T looks like

$$\begin{bmatrix} J_{a_1}(\lambda_1) & 0 & 0 & \dots & 0 \\ 0 & J_{a_2}(\lambda_2) & 0 & \dots & 0 \\ 0 & 0 & J_{a_3}(\lambda_3) & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & J_{a_s}(\lambda_s) \end{bmatrix},$$

where $J_a(\lambda)$ is the $a \times a$ matrix

$$J_a(\lambda) = \begin{bmatrix} \lambda & 0 & \dots & \dots & 0 \\ 1 & \lambda & 0 & \ddots & \vdots \\ 0 & 1 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & \lambda \end{bmatrix}.$$

The Jordan canonical form of T is unique (up to rearranging order of the Jordan blocks $J_{a_i}(\lambda_i)$. At any rate, T is diagonalizable if and only if each a_i is 1. (I have come upon elementary linear algebra books that say a square matrix over $\mathbb{C}$ is "defective" if it isn't diagonalizable. I scratch my head and think, "The matrix isn't defective; it merely has a more complicated Jordan canonical form than simply being diagonalizable.) I emphasize that there is nothing numerical about the theory of canonical forms of matrices. One is merely describing the structure of modules over Principal Ideal Domains; in other words, this result is exactly the same as the result which gives the structure of finitely generated Abelian groups. I have to point out how the data $(\boldsymbol{k}, V, T)$ involves a module over a PID. One says V is a module over the polynomial ring $\boldsymbol{k}[x]$, where the scalar multiplication xv is given by $xv = T(v)$ for all $v \in V$.

3. Galois Theory.

1.1. Let $f(x)$ be a polynomial with rational coefficients and let F be the smallest subfield of $\mathbb{C}$ which contains $\mathbb{Q}$ and the roots of f. We associate a group G to the pair of fields $\mathbb{Q} \subseteq F$. We prove the following statements.

(a) The group G is finite and the number of elements in G is equal to the dimension of F as a vector space over $\mathbb{Q}$.

(b) There is a one-to-one correspondence between the subgroups of G and the intermediate fields K with $\mathbb{Q} \subseteq K \subseteq F$. (The Sylow theorems give us information about the subgroups of a group!)

(c) The polynomial f is **solvable by radical** if and only if G is solvable.

(d) There are fifth degree polynomials which are not solvable by radical. In other words, **it is not possible to give a formula for the solutions of a fifth degree polynomial equation in terms of a finite iteration of taking roots and doing addition, subtraction, multiplication, and division.** Of course, the quadratic formula gives the solutions of a quadratic polynomial equation. The course will include the formulas for finding the solutions of third and fourth degree polynomial equations.

1.I. **Some of the reasons that I really like Algebra.**

(a) In algebra, one states up front what the rules are.

(b) Algebra is not confined to studying something that "is already there". That is, one can change the rules. For example, one can decree, "Today, two is equal to zero." Henceforth, now one has $(x + y)^2 = x^2 + y^2$.

(c) In algebra, the words are well-defined.

(d) Algebra provides tools for proving statements and making calculations. Here are some examples.

  (i) One often calculates the multiplicity of an intersection by calculating the length of a local ring.

  (ii) One often proves that two topological spaces are not homeomorphic by showing that some algebraic invariant of the spaces are different.

  (iii) Algebra provides interesting things for combinatorists to count; and algebra provides new techniques for counting things.

2. Groups.

2.A. The definition and elementary properties of groups.

Definition 2.1. A group is a set G together with a function $G \times G \to G$, given by $(g_1, g_2) \mapsto g_1 * g_2$, for $g_i \in G$, which satisfies the following properties.

(a) If g_1, g_2, and g_3 are elements of G, then $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$.
(b) There is an element e in G with $e * g = g$ and $g * e = g$ for all g in G.
(c) For each g in G, there exists an element g' with $g * g' = e$ and $g' * g = e$.

If G also satisfies $g_1 * g_2 = g_2 * g_1$ for all g_i in G, then G is called an Abelian group.

Remarks 2.2. (a) The function $*$ of is usually called an "operation" on G.
(b) One emphasizes that $g_1 * g_2$ is in G for all pairs of elements of G by saying that "G is closed under the operation $*$".
(c) Property (a) is called the associative property of the group $(G, *)$.
(d) The element e of (b) is called an "identity" element of the group $(G, *)$.
(e) The element g' of (c) is called an "inverse" of g.
(f) If $(G, *)$ is a group and H is a non-empty subset of G which is closed under $*$ and closed under the process of taking inverses, then $(H, *)$ is also a group and $(H, *)$ is called a subgroup of G.

Observation 2.3. *Let $(G, *)$ be a group. Then the following statements hold.*

(a) *The identity element of G is unique.*
(b) *If g is an element of G, then the inverse of g is unique.*
(c) *If g is an element of G and g' is the inverse of g, then g is the inverse of g'.*

Proof.
(a) If e and e_0 both are identity elements of G, then

$$e_0 = e * e_0 = e.$$

The equality on the left holds because e is an identity element of G. The equality on the right holds because e_0 is an identity element of G.
(b) If h and h_0 both are inverses of the element g of G, then

$$\begin{aligned} h_0 &= e * h_0, && \text{because } e \text{ is the identity element of } (G,*),\\ &= (h * g) * h_0, && \text{because } h \text{ is an inverse of } g,\\ &= h * (g * h_0), && \text{because } * \text{ is an associative operation on } G,\\ &= h * e, && \text{because } h_0 \text{ is an inverse of } g,\\ &= h, && \text{because } e \text{ is the identity element of } (G,*). \end{aligned}$$

(c) The hypothesis that g' is the inverse of g guarantees that $g' * g = e$ and $g * g' = e$. These two equations also demonstrate that g acts like an inverse of g'. Apply (b) to see that the inverse of g' in G is unique. It follows that g is the inverse of g'. □

2.B. **Examples of groups.**

Example 2.4. The set of integers under addition is an Abelian group, denoted $(\mathbb{Z}, +)$.

Example 2.5. The set of non-zero complex numbers under multiplication is an Abelian group, denoted $\mathbb{C}^* = (\mathbb{C} \setminus \{0\}, \times)$.

Example 2.6. The set of invertible $n \times n$ matrices with complex entries under multiplication is a non-Abelian group, denoted $\mathrm{GL}_n(\mathbb{C})$. (The symbol GL stands for General Linear.)

Example 2.7. The set of $n \times n$ matrices with complex entries and determinant one is a non-Abelian group, denoted $\mathrm{SL}_n(\mathbb{C})$. (The symbol SL stands for Special Linear.)

Example 2.8. The set of permutations of the set $\{1, 2, 3, \dots, n\}$ under composition forms a group, denoted S_n.

We study S_3 in more detail. First we will list the elements of S_3 using two-rowed notation. Then we will list the elements of S_3 using one-rowed notation. (It is perfectly obvious what the two-rowed notation means; but it takes much too much effort to write an element down. One must think about what one-rowed notation means; but it clearly is more convenient to use.)

The notation

$$\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$$

means that $1 \mapsto a$, $2 \mapsto b$, and $3 \mapsto c$. The elements of S_3 are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

Instead of two-rowed notation we use cycle notation (or one-rowed) notation:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1), \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1,3,2), \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1,2,3),$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1,2), \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1,3), \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (2,3).$$

The cycle $(1,2,3)$ represents

$$\begin{tikzcd} 1 \arrow[rr] & & 2 \arrow[dl] \\ & 3 \arrow[ul] & \end{tikzcd}$$

Compose permutations the same way you always compose functions; in other words,

$$(f \circ g)(x) = f(g(x)).$$

In particular,

$$(1,2) \qquad \underbrace{(1,3)}_{\text{Apply this function first}} \qquad = (1,3,2)$$

If it is necessary, think in the two-rowed language

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

In a similar manner

$$(1,3)(1,2) = (1,2,3).$$

We are about to record the multiplication table for S_3. Take $\sigma = (1,2)$ and $\rho = (1,2,3)$. We should verify that all six permutations of $\{1,2,3\}$ can be expressed in the form $\sigma^i\rho^j$ with $0 \le i \le 1$ and $0 \le j \le 2$. Also, record the multiplication table for S_3.

Exercise. Here is a small problem. Let $\sigma = (1,2)$ and $\rho = (1,2,3)$.

(i) Show that

$$S_3 = \{\sigma^i\rho^j|\,|\,0 \le i \le 1 \text{ and } 0 \le j \le 2\}$$

(ii) Fill in the multiplication table

	id	ρ	ρ^2	σ	$\sigma\rho$	$\sigma\rho^2$
id						
ρ						
ρ^2						
σ						
$\sigma\rho$						
$\sigma\rho^2$						

Answers. Here are my answers:

$$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} = (1) = \sigma^0,\ \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} = (1,3,2) = \rho^2,\ \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} = (1,2,3) = \rho,$$

$$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} = (1,2) = \sigma,\ \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} = (1,3) = \sigma\rho,\ \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = (2,3) = \sigma\rho^2.$$

	id	ρ	ρ^2	σ	$\sigma\rho$	$\sigma\rho^2$
id	id	ρ	ρ^2	σ	$\sigma\rho$	$\sigma\rho^2$
ρ	ρ	ρ^2	id	$\sigma\rho^2$	σ	$\sigma\rho$
ρ^2	ρ^2	id	ρ	$\sigma\rho$	$\sigma\rho^2$	σ
σ	σ	$\sigma\rho$	$\sigma\rho^2$	id	ρ	ρ^2
$\sigma\rho$	$\sigma\rho$	$\sigma\rho^2$	σ	ρ^2	id	ρ
$\sigma\rho^2$	$\sigma\rho^2$	σ	$\sigma\rho$	ρ	ρ^2	id

Example 2.9. Let G be the set of rotations of the xy-plane which fix the origin. It is easy to see that G is an Abelian group.

Example 2.10. Let $\mathscr{G}$ be

$$\{\text{rotations of the } xy\text{-plane which fix the origin}\}$$
$$\cup\ \{\text{reflections of the } xy\text{-plane across a line through the origin}\}.$$

In Homework problem 1, you will show that G is a group. I propose that you use the following technique. Let $\begin{bmatrix}x\\y\end{bmatrix}$ represent the vector which joins origin to the point (x,y) in the xy-plane. Let

$f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function which fixes the origin and rotates each vector in $\mathbb{R}^2$ by the angle θ. Find the matrix M with the property that

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = M \begin{bmatrix} x \\ y \end{bmatrix}.$$

Let ℓ be the line through the origin which passes through the origin and makes the angle θ_1 with the positive x-axis and let $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the function which reflects each vector $\begin{bmatrix} x \\ y \end{bmatrix}$ across ℓ. Find the matrix M_1 with the property that

$$f_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = M_1 \begin{bmatrix} x \\ y \end{bmatrix}.$$

As you examine the matrices, it will become obvious that G is closed under composition; and hence is a group. (It turns out that the matrices that arise in Example 2.9 form the group which is called the Special Orthogonal group $\mathrm{SO}_2(\mathbb{R})$ and the matrices that arise in Example 2.10 form the group which is called the Orthogonal group $\mathrm{O}_2(\mathbb{R})$. (The matrix M of $\mathrm{GL}_n(\mathbb{R})$ is in $\mathrm{O}_n(\mathbb{R})$ if $M^{-1} = M^{\mathrm{T}}$. The matrix M of $\mathrm{O}_n(\mathbb{R})$ is in $\mathrm{SO}_n(\mathbb{R})$ if $\det M = 1$.) [2]

Definition. Let n be an integer with $3 \le n$. For each non-negative j, let v_j be the point $(\cos \frac{2\pi j}{n}, \sin \frac{2\pi j}{n})$ in the xy-plane. The standard regular n-gon is the polygon with vertices $\{v_j \mid 0 \le j \le n-1\}$ and edges

$$\{\text{the line segment which join } v_j \text{ to } v_{j+1} \mid 0 \le j \le n-1\}.$$

Example 2.11. Let $\mathscr{G}$ be the group of Example 2.10. Fix an integer n with $3 \le n$. Let D_n be the subgroup of $\mathscr{G}$ which carries the regular n-gon onto itself. (The group D_n is called the n^{th} Dihedral group.) The regular n-gon has n sides of equal length, center at $(0,0)$, and one vertex at $(1,0)$.

Example. In particular, the regular 3-gon has vertices $(1,0)$, $(-1/2, \sqrt{3}/2)$, and $(-1/2, -\sqrt{3}/2)$. Label these vertices 1, 2, 3. The elements of D_3 are the identity (this is the permutation (1) of the vertices), rotation by $2\pi/3$ radians (this is the permutation $(1,2,3)$ of the vertices), rotation by $4\pi/3$ radians (this is the permutation $(1,3,2)$ of the vertices), reflection across the x-axis (this is the permutation $(2,3)$ of the vertices), reflection across the line through the origin and vertex 2 (this is the permutation $(1,3)$ of the vertices), and reflection across the line through the origin and vertex 3 (this is the permutation $(1,2)$ of the vertices). Observe that D_3 is equal to S_3.

Theorem 2.11.1. *Let ρ be rotation by $\frac{2\pi}{n}$ and σ be reflection across the x-axis. Then every element of D_n can be written uniquely in the form*

$$\sigma^i \rho^j \textit{ with } 0 \le i \le 1 \textit{ and } 0 \le j \le n-1.$$

In particular, D_n has $2n$ elements.

Proof.
Part one. We show that every element of D_n can be written in the given form.

[2] In general we will write M^{T} for the transpose of the matrix M. If $m_{i,j}$ is in row i, column j of M, then $m_{i,j}$ is in row j, column i of M^{T}.

It is clear that the only rotations from $\mathscr{G}$ that carry the n-gon to it self are ρ^j for $0 \le j \le n-1$.

Suppose τ is a reflection from $\mathscr{G}$ and τ carries the n-gon to itself. You will show in Homework 1, that $\sigma\tau$ is a rotation. Thus, $\sigma\tau = \rho^j$ for some j. Multiply both sides of the equation on the left by σ to see that $\tau = \sigma\rho^j$.

Part two. We show uniqueness.

Suppose $\sigma^{i'}\rho^{j'} = \sigma^i\rho^j$ with $j' \le j$. Then

$$\underbrace{\sigma^{0\text{ or }1}}_{\text{We fix vertex 1}} = \rho^{j-j'}.$$

If $j - j'$ is positive, then the right side moves every vertex. Thus, $j - j' = 0$ and hence $i = i'$. □

Example. The group D_4 consists of the identity map, three rotations, and reflection across the x-axis, $\underbrace{y = x}_{\ell_1}$, the $\underbrace{y\text{-axis}}_{\ell_2}$, and $\underbrace{y = -x}_{\ell_3}$. DRAW A PICTURE. In Homework 2, among other things, you will write the reflections across ℓ_1, ℓ_2, and ℓ_3 in the form of Theorem 2.11.1.

Example 2.12. Return to the group $\mathbb{C}^* = (\mathbb{C} \setminus \{0\}, \times)$ of Example 2.5.

Facts-Definitions

(a) If $z = x + \imath y$ with $x, y \in \mathbb{R}$, then $z = r(\cos\theta + \imath\sin\theta)$ with r and θ in $\mathbb{R}$. DRAW A PICTURE.

(b) If z is the complex number of (a), then $|z| = \sqrt{x^2 + y^2} = |r|$.

(c) If r_1, r_2, θ_1, and θ_2 are real numbers, then

$$r_1(\cos\theta_1 + \imath\sin\theta_1) \cdot r_2(\cos\theta_2 + \imath\sin\theta_2) = r_1r_2(\cos(\theta_1+\theta_2) + \imath\sin(\theta_1+\theta_2)).$$

(d) If $\theta \in \mathbb{R}$, then $\cos\theta + \imath\sin\theta = e^{\imath\theta}$.

My favorite way to think of these "facts" is through Taylor's series from calculus. Recall, from calculus, that the following equations hold for all real numbers x:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \\ \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \text{and} \\ \cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}. \end{aligned}$$

The Taylor series for the exponential function continues to hold if x is replaced by a complex number. Thus, if θ is a real number then

$$\begin{aligned} e^{\imath\theta} &= 1 + (\imath\theta) + \frac{(\imath\theta)^2}{2!} + \frac{(\imath\theta)^3}{3!} + \frac{(\imath\theta)^4}{4!} + \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \dots\right) \\ &= \cos\theta + \imath\sin\theta. \end{aligned}$$

This "explains" (d).

If $z = r(\cos\theta + \mathbb{i}\sin\theta)$ and one believes (d), then (a) yields that $z = re^{\mathbb{i}\theta}$.

If $z = re^{\mathbb{i}\theta}$, then (b) yields $|z| = |r|$ and $|e^{\mathbb{i}\theta}| = 1$. (The non-negative real number $|z|$ is called the modulus of z. If $z = a + b\mathbb{i}$ with a and b real, then $|z| = \sqrt{a^2+b^2} = \sqrt{\bar{z}z}$, where $\bar{z} = a - b\mathbb{i}$, is the complex conjugate of z.)

If $z_j = r_j e^{\mathbb{i}\theta_j}$, then (c) becomes

$$r_1 e^{\mathbb{i}\theta_1} \cdot r_2 e^{\mathbb{i}\theta_2} = r_1 r_2 e^{\mathbb{i}(\theta_1+\theta_2)}$$

Example 2.12.1. Let U be the subgroup of $\mathbb{C}^*$ consisting of all elements of modulus 1. Then U is a group.

Example 2.12.2. Some finite subgroups of $\mathbb{C}^*$ are $\{1\}$, $\{1,-1\}$, $\{1,\mathbb{i},-1,-\mathbb{i}\}$. Homework problem 3 asks you to find all finite subgroups of $\mathbb{C}^*$.

Example 2.13. Let I be an index set. Suppose that for each $i \in I$, G_i is a group. The direct product of $\{G_i \mid i \in I\}$ is the group

$$\prod_{i\in I}(G_i, *_i) = \{(g_i)_{i\in I} | g_i \in I\}.$$

The operation in the direct product is given component-wise. That is, the i-tuple $(g_i)_{i\in I}$ times the i-tuple $(g_i')_{i\in I}$ is equal to the i-tuple

$$(g_i *_i g_i')_{i\in I}.$$

The direct sum of the G_i is the group

$$\bigoplus_{i\in I} G_i = \{(g_i)_{i\in I} | g_i \in I \text{ and at most finitely many } g_i \text{ are not } 1\}.$$

The operation in the direct sum is also given component-wise. Of course, if $I = \{1,\dots,n\}$, then

$$G_1 \oplus G_2 \oplus \dots \oplus G_n = G_1 \times G_2 \times \cdots \times G_n.$$

Let C_2 be the group with two elements. In Homework problem 4, I have asked you to count the number of four element subgroups of $C_2 \times C_2 \times C_2 \times C_2$.

Direct product and direct sum satisfy the following universal mapping properties. (One could define direct sum and direct product by way of these UMPs.)

Observation. *Let I be an index set. Suppose that for each $i \in I$, G_i is a group.*

(a) *Let G be a group and, for each i, let $\phi_i : G \to G_i$ be a group homomorphism. Then there exists a unique group homomorphism $\Phi : G \to \prod_{i\in I} G_i$ so that the diagram*

$$\begin{tikzcd} G \arrow[r, dashed, "\exists!\Phi"] \arrow[dr, "\phi_{i_0}"'] & \prod_{i\in I} G_i \arrow[d, "\mathrm{proj}_{i_0}"] \\ & G_{i_0} \end{tikzcd}$$

commutes for all $i_0 \in I$.

(b) *Let G be an Abelian group and, for each i, let $\phi_i : G_i \to G$ be a group homomorphism. Then there exists a unique group homomorphism $\Phi : \bigoplus_{i\in I} G_i \to G$ so that the diagram*

$$\begin{tikzcd}
G & \bigoplus_{i\in I} G_i \arrow[l, dashed, "\exists!\Phi"'] \\
& G_{i_0} \arrow[ul, "\phi_{i_0}"] \arrow[u, "\mathrm{incl}_{i_0}"']
\end{tikzcd}$$

commutes for all $i_0 \in I$.

Note. You should write down complete proofs for these small facts.

Definition 2.14. If G and G' are groups then a function $\phi : G \to G'$ is a group homomorphism if

$$\phi(g_1g_2) = \phi(g_1)\phi(g_2),$$

for all g_1, g_2 in G. The operation on the left takes place in G. the operation on the right takes place in G'.

Elementary Properties. Let $\phi : G \to G'$ be a group homomorphism. The following statements hold.

- If e is the identity element of G, then $\phi(e)$ is the identity element of G'.
- The homomorphism ϕ carries the inverse of g to the inverse of $\phi(g)$ for all $g \in G$.

Note. You should write down complete proofs for these small facts.

Last time we said that if G and G' are groups, then a function $\phi : G \to G'$ is a group homomorphism if

$$\phi(g_1 * g_2) = \phi(g_1) * \phi(g_2)$$

for all $g_1, g_2 \in G$.

A group homomorphism which is one-to-one and onto is called a group isomorphism.

Example 2.14.1. The function $\phi : (\mathbb{R}, +) \to (\{r \in \mathbb{R} \mid 0 < r\}, \times)$, given by $\phi(r) = e^r$ is a group isomorphism because

$$\phi(r_1 + r_2) = e^{r_1+r_2} = e^{r_1}e^{r_2} = \phi(r_1)\phi(r_2).$$

ϕ is surjective. Take s in the target. Observe that $\ln s$ is in the source and $\phi(\ln s) = e^{\ln s} = s$.

ϕ is injective. Suppose r_1 and r_2 are in the source with $\phi(r_1) = \phi(r_2)$. Then $e^{r_1} = e^{e_2}$. Apply ln to both sides to learn that $r_1 = r_2$.

Example 2.14.2. The function $\phi : (\mathbb{R}, +) \to U$, which is given by

$$\phi(\theta) = e^{\imath\theta}$$

is a surjective group homomorphism which is not injective.

Example 2.14.3. The function $\phi : U \to \mathrm{SO}_2(\mathbb{R})$, which is given by

$$\phi(e^{i\theta}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

is a group isomorphism. The most interesting part of this claim is making sure that "ϕ is well-defined". That is, one must show that if $e^{i\theta} = e^{i\theta'}$, then

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{bmatrix}.$$

Example 2.15. Let G be the collection of rotations of $\mathbb{R}^3$ which fix a line through the origin.

Claim 2.15.1. *The set G forms a group.*

Notice that every element of G is given by matrix multiplication. Indeed, if f is rotation about the z-axis, then f is rotation of the xy-plane and you will show in Homework 1 that f is given by matrix multiplication. Observe that rotation about the line ℓ through the origin is the composition

$$\text{(move the } z\text{-axis to } \ell)\circ(\text{rotate about the } z\text{-axis})\circ(\text{move } \ell \text{ to the } z\text{-axis}) \tag{2.15.2}$$

Each of the three functions in (2.15.2) is given by matrix multiplication[3] and the composition is given by the product of the three matrices.

Claim 2.15.3. *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a function and M be a matrix in $\mathrm{GL}_3(\mathbb{R})$ with*

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = M\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

for all

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3.$$

Then

$$f \in G \iff M \in \mathrm{SO}_3(\mathbb{R}).$$

Observe that Claim 2.15.3 implies Claim 2.15.1. We prove Claim 2.15.3.

(Recall that a 3 × 3 matrix M with real entries is in $\mathrm{SO}_3(\mathbb{R})$ if and only if MM^{T} is the 3 × 3 identity matrix and $\det M = 1$.)

Proof. ($\Rightarrow$)[4] Let ℓ be a line through the origin in $\mathbb{R}^3$ and f be rotation which fixes ℓ. Let

$$v_1, v_2, v_3 \text{ be an orthonormal basis for } \mathbb{R}^3 \text{ with } v_3 \text{ on } \ell. \tag{2.15.4}$$

Recall the following statements.

(a) The vectors v_1, v_2, v_3 are an orthonormal basis for $\mathbb{R}^3$ if $v_i^{\mathrm{T}} v_j$ is equal to the Kronecker delta, for $1 \le i, j \le 3$.[5]

[3] When we "move ℓ to the z-axis" we want to do this in a systematic manner!

[4] In Homework problem 6 you will carry out this procedure for some explicit specific data.

[5] We use $(-)^{\mathrm{T}}$ to mean "transpose".

(b) One way to get vectors as described in (2.15.4) is to start with a unit vector v_3 on ℓ, extend v_3 to be a basis of $\mathbb{R}^3$, and then apply Gram-Schmidt orthogonalization.

We compute the matrix for f as described in (2.15.2). One good matrix for moving the z-axis to ℓ is

$$Q = \left[\ v_1 \mid v_2 \mid v_3\ \right].$$

The matrix Q sends the x-axis to the line containing v_1 and the origin. The matrix Q also sends the y-axis to the line containing v_2 and the origin. The inverse of Q sends ℓ to the z-axis. Of course, the columns of Q are an orthonormal set; so the inverse of Q is Q^{T}. According to Homework problem 1, the matrix for rotation around the z-axis has the form

$$N = \left[\begin{array}{c|c} M' & 0 \\ \hline 0 & 1 \end{array}\right],$$

for some M' in $\mathrm{SO}_2(\mathbb{R})$. Thus, the matrix for f is

$$M = QNQ^{\mathrm{T}}.$$

Observe that M is an orthogonal matrix because

$$M^{\mathrm{T}} = (Q^{\mathrm{T}})^{\mathrm{T}} N^{\mathrm{T}} Q^{\mathrm{T}} = QNQ^{\mathrm{T}} = M.$$

Furthermore, $\det N = 1$ and $\det Q = \det Q^{\mathrm{T}}$ with $\det Q \det Q^{\mathrm{T}} = \det I = 1$. Thus, $\det Q = \det Q^{\mathrm{T}} = \pm 1$ and $\det M = 1$. We have proven that $M \in \mathrm{SO}_3(\mathbb{R})$.

($\Leftarrow$) Let M be an element of $\mathrm{SO}_3(\mathbb{R})$. We will show that there exists a line ℓ through the origin and an angle θ so that Mv is the vector that is obtained by rotating v about ℓ by the angle θ, for all v in $\mathbb{R}^3$.

Theorem. *Let G be the following set of functions together with the operation composition*:

$$G = \{f : \mathbb{R}^3 \to \mathbb{R}^3 \mid f \textit{ is a rotation of } \mathbb{R}^3 \textit{ which fixes a line through the origin}\}.$$

Then

(1) *Each element of G is given by matrix multiplication.*
(2) *The set $(G, \circ)$ is a group.*
(3) *The groups $(G, \circ)$ and $\mathrm{SO}_3(\mathbb{R})$ are isomorphic.*

The proposed isomorphism is

$$\mathrm{SO}_3(\mathbb{R}) \to G$$

is the function which sends $M \in \mathrm{SO}_3(\mathbb{R})$ to the function $\mathrm{mult}_M : \mathbb{R}^3 \to \mathbb{R}^3$, where

$$\mathrm{mult}_M(v) = Mv,$$

for $v \in \mathbb{R}^3$. (The vector space $\mathbb{R}^3$ is the Abelian group of column vectors with three real entries.)

Last time we established (1) and we showed that if $f \in G$ is multiplication by M, then M is an element of $\mathrm{SO}_3(\mathbb{R})$.

Today we show that if $M \in \mathrm{SO}_3(\mathbb{R})$, then the function $\mathbb{R}^3 \to \mathbb{R}^3$ which is given $v \mapsto Mv$ is rotation by some angle about a line through the origin.

In particular, we must show that if M is in $\mathrm{SO}_3(\mathbb{R})$, then M fixes some non-zero vector v and M carries the plane perpendicular to v to itself.

In Homework problem 5, you will prove that M is diagonalizable over $\mathbb{C}$. In other words, you will prove that there are complex numbers λ_1, λ_2 and λ_3 and linearly independent vectors v_1, v_2, and v_3 in $\mathbb{C}^3$ such that $Mv_i = \lambda_i v_i$, for $1 \le i \le 3$. Observe that

(a) If λ is an eigenvalue of M, then $\bar\lambda$ is an eigenvalue of M.
(b) $\prod \lambda_i = 1$
(c) $|\lambda_i| = 1$ for $1 \le i \le 3$.
(d) the product $(1 - \bar\lambda_i\lambda_j)\bar v_i^{\mathrm{T}} v_j$ is zero for all i and j.

For (a), notice that if $Mv = \lambda v$, then $\bar M \bar v = \bar\lambda\bar v$; but M has real entries, so $\bar M = M$ and therefore, $M\bar v = \bar\lambda \bar v$. Conclude that $\bar\lambda$ is an eigenvalue of M.

For (b), the matrix M is similar to the diagonal matrix with the eigenvalues on the main diagonal. Similar matrices have the same determinant. The determinant of M is 1. It follows that $\prod \lambda_i = 1$.

For (c), observe that if $Mv = \lambda v$, with $v \ne 0$, then

$$\bar v^{\mathrm{T}} v = \bar v^{\mathrm{T}} \bar M^{\mathrm{T}} M v = (\overline{Mv})^{\mathrm{T}} M v = (\overline{\lambda v})^{\mathrm{T}} \lambda v = \bar\lambda\lambda \bar v^{\mathrm{T}} v.$$

The complex number $\bar v^{\mathrm{T}} v$ is not zero; hence the complex number $\bar\lambda\lambda$ must be 1. In other words, the modulus of λ must be 1.[6]

For (d)[7], observe that

$$\bar v_i^{\mathrm{T}} v_j = \bar v_i^{\mathrm{T}} \bar M^{\mathrm{T}} M v_j = (\overline{Mv_i})^{\mathrm{T}} M v_j = (\overline{\lambda_i v_i})^{\mathrm{T}} \lambda_j v_j = \bar\lambda_i \lambda_j \bar v_i^{\mathrm{T}} v_j.$$

It follows immediately from (a)–(c) that at least one of the eigenvalues of M is real; hence the eigenvalues of M are:

$$1, 1, 1 \quad \text{or} \quad 1, -1, -1 \quad \text{or} \quad 1, a + bi, a - bi$$

with b not zero and $\sqrt{a^2 + b^2} = 1$.

If the eigenvalues of M are $1, 1, 1$, then M is the identity matrix which is rotation fixing the z-axis by angle zero.

If the eigenvalues of M are $1, -1, -1$, (and M is diagonalizable) then there are linearly independent vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ with $Mv_i = \lambda_i v_i$ with $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -1$.[8] Apply (d) to see that v_1 is perpendicular to both v_2 and v_3. Observe that M is the matrix for rotation by π about the line containing v_1.

Now we focus on the case where the eigenvalues of M are 1, $a + bi$, and $a - bi$ with a and b in $\mathbb{R}$ with $a^2 + b^2 = 1$ and b not zero. Let $u_1 + iu_2$ be a non-zero eigenvector of M associated to 1, $w_1 + iw_2$ be a non-zero eigenvector of M associated to $a + bi$, and $w_3 + iw_4$ be a non-zero eigenvector of M associated to $a - bi$ with $u_1, u_2, w_1, w_2, w_3, w_4 \in \mathbb{R}^3$. It is clear that u_1 and u_2

[6] If $\lambda = a + bi$ is a complex number with a and b real, then $\sqrt{a^2 + b^2}$ is called the modulus of λ and is denoted $|\lambda|$.

[7] Of course, one can prove (c) and (d) simultaneously.

[8] The fact that there exist linearly independent $v_1, v_2, v_3 \in \mathbb{R}^3$ with $Mv_i = \lambda_i v_i$ requires a small amount of argument. At first, we are guaranteed linearly independent w_1, w_2, w_3 in $\mathbb{C}^3$ with $M_i w_i = \lambda_i w_i$ for $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -1$. But $w_i = a_i + ib_i$ with $a_i, b_i \in \mathbb{R}^3$. The vectors a_i and b_i are necessarily eigenvectors of M associated to λ_i. It is possible to pick v_1 from the set $\{a_1, b_1\}$ and v_2, v_3 from the set $\{a_2, b_2, a_3, b_3\}$.

each are eigenvectors of M belonging to 1. At least one of the vectors u_1 and u_2 is non-zero; we have identified a non-zero vector $v_1 \in \mathbb{R}^3$ with $Mv_1 = v_1$. (We may as well insist that v_1 is a unit vector.) According to (d), the vectors w_1, w_2, w_3, and w_4 of $\mathbb{R}^3$ all are in the plane perpendicular to v_1. The vectors $w_1 + i w_2$ and $w_3 + i w_4$ span a two dimensional subspace of of $\mathbb{C}_2$; so the vectors w_1, w_2, w_3, and w_4 of $\mathbb{R}^3$ can not lie on a line; they must span the plane in $\mathbb{R}^3$ perpendicular to v_1. Pick an orthogonal set v_1, v_2, v_3. Noticed that the vector spaces (w_1, w_2, w_3, w_4) and (v_2, v_3) are equal. The hypothesis that $w_1 + i w_2$ and $w_3 + i w_4$ are eigenvectors of M ensure that Mv_2 and Mv_3 are both in (v_2, v_3). The matrix for M with respect to the basis v_1, v_2, v_3 has the form

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & M' \end{array}\right].$$

The matrix M is in $\mathrm{SO}_3(\mathbb{R})$; hence, the matrix M' is in $\mathrm{SO}_2(\mathbb{R})$. You proved in Homework 1, that the matrices of $\mathrm{SO}_2(R)$ are rotation matrices. Thus, multiplication by M fixes the line containing v_1 and rotates the plane perpendicular to v_1. □

2.C. **Cayley's Theorem.** What you should take away from HW2:

- problem 3: For each positive integer n, there is exactly one subgroup of $\mathbb{C}^*$ of order n, namely $U_n = \{e^{\frac{2\pi i j}{n}} \mid 0 \le j \le n-1\}$. This is the group of n^{th} roots of 1 in $\mathbb{C}$.
- problem 4: Each element in a direct sum is interesting. (If G_1 and G_2 are groups, then the element (g_1, g_2) is just interesting as the elements (g_1, id_{G_2}) and (id_{G_1}, g_2), where $g_i \in G_i$ and id_{G_i} is the identity element of G_i.)

  If you recognized that "each element in a direct sum is interesting", but did not count well, then work on your counting skills. (One way to do this is to teach Math 574 or 374 ...).
- problem 5: Wow. There is so much to learn from problem 5.
  - The $n \times n$ matrix M with entries in the field F is diagonalizable if and only if F^n has a basis of eigenvectors. (I did not know that you wouldn't know this. In fact, I introduced the expression "diagonalizable" by accident. I wanted you to prove that if $M \in SO_3(\mathbb{R})$, then $\mathbb{R}^3$ has a basis of eigenvectors. I swapped the desired condition for an equivalent but irrelevant condition without realizing that I had done it.) Nonetheless, many folks taught themselves this result and wrote down a proof. Excellent!
  - It is good to understand the assertion that if V is subspace of F^n, F is an algebraically closed field, M is an $n \times n$ matrix with entries in F, and $Mv \in V$ for all $v \in V$, then M has an eigenvector in V.
  - It is worth your while to know the concept "diagonalizable" because our unit on canonical forms of matrices answers the question "Well, if a square matrix is not diagonalizable, why isn't it and what is it."
  - By the way nilpotent matrices are not diagonalizable:

    $$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

    and the sum of a diagonal matrix and a nilpotent matrix is not diagonalizable

$$\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

- Let M be an $n\times n$ matrix over the field F. It is very much the philosophy of our course to consider subspaces V of F^n with $Mv \in V$ for all $v \in V$. (This makes V an $F[M]$-module or an $F[x]$-module where xv is defined to be Mv.) If v is an eigenvector of M then Fv is a one-dimensional vector space which is an $F[M]$-module.
- Orthogonal matrices over $\mathbb{R}$ preserve length and angle. (I had not stated this explicitly; indeed, I had not thought it explicitly; but I was asked about it after class the other day and it is such a good question, that I want to share it with every body). If M is an $n\times n$ matrix with real entries and $MM^{\mathrm{T}} = I = M^{\mathrm{T}}M$, then $(Mv)^{\mathrm{T}}(Mv) = v^{\mathrm{T}}v$; so v and Mv have the same length for each $v \in \mathbb{R}^n$. Furthermore $(Mv)^{\mathrm{T}}Mw = v^{\mathrm{T}}w$ for all v and w in $\mathbb{R}^n$. So, the angle between v and w is the same as the angle between Mv and Mw. (Recall that the dot product of two vectors is the length of the first vector times the length of the second vector times the cosine of the angle between them.)
- One proves the result of problem 5 by induction. My argument decomposes an $F[M]$-module as a direct sum of two smaller $F[M]$-modules. This argument is also very much in keeping with the philosophy of our course.

- Problem 6. If one has to make a dirty calculation, then make it all the way to the bitter end and then clean it up. Now one has a chance of making sure that one has the correct answer. This philosophy comes into play when I am teaching and when I am writing research papers.

My answer to number 5:

Theorem. *Every unitary matrix from* $\mathrm{GL}_n(\mathbb{C})$ *is diagonalizable.*

The proof is a consequence of the following Claim; just take V to be all of $\mathbb{C}^n$.

Claim. *Let M be a unitary $n \times n$ matrix with complex entries and let V be a subspace of $\mathbb{C}^n$ with the property that $MV \subseteq V$. Then the restriction of M to V is diagonalizable.*

Proof. Write $M|_V$ for the "restriction of M to V".

We prove the claim by induction on the dimension of V. If $\dim V = 1$, and v is a non-zero element of V, then v is a basis for V. The hypothesis that $MV \subseteq V$ guarantees that v is an eigenvalue of M and hence $M|_V$ is diagonalizable.

Now suppose that $1 < \dim V$. Recall that $M|_V$ has a non-zero eigenvector. [9]

Let v_0 be a non-zero eigenvector of $M|_V$ which belongs to the eigenvalue λ_0. (The matrix M is non-singular, so $\lambda_0 \neq 0$.) Let

$$W = \{w \in V \mid \bar{w}^{\mathrm{T}}v_0 = 0\}.$$

It is clear that W is a vector space. Observe that

- $V = \mathbb{C}v_0 \oplus W$, and

[9] Indeed, the characteristic polynomial of $M|_V$ is a polynomial in one variable with complex coefficients. Such a polynomial has a root, say λ_1 in $\mathbb{C}$ (by the "Fundamental Theorem of Algebra"). Thus $M|_V - \lambda_1\,\mathrm{id}$ is a singular matrix. Any non-zero vector in the null space of $M|_V - \lambda_1\,\mathrm{id}$ is an eigenvector of $M|_V$.

- $MW \subseteq W$.

Once we are confident with these assertions then the proof of the claim is complete by induction because $\dim W < \dim V$. We establish the two assertions.

We first show that $MW \subseteq W$. If $w \in W$, then $\bar{w}^{\mathrm{T}} v_0 = 0$ and

$$(\overline{Mw})^{\mathrm{T}} v_0 = \frac{1}{\lambda_0}(\overline{Mw})^{\mathrm{T}} M v_0 = \frac{1}{\lambda_0}\bar{w}^{\mathrm{T}}\overline{M}^{\mathrm{T}} M v_0 = \frac{1}{\lambda_0}\bar{w}^{\mathrm{T}} \operatorname{id} v_0 = \bar{w}^{\mathrm{T}} v_0 = 0;$$

hence, Mw is in W, as claimed.

Now we show that V is contained in the sum of W and $\mathbb{C}v_0$. If v is an arbitrary element of V, then

$$v = (v - \frac{\bar{v}^{\mathrm{T}} v_0}{\bar{v}_0^{\mathrm{T}} v_0} \cdot v_0) + \frac{\bar{v}^{\mathrm{T}} v_0}{\bar{v}_0^{\mathrm{T}} v_0} \cdot v_0$$

with $(v - \frac{\bar{v}^{\mathrm{T}} v_0}{\bar{v}_0^{\mathrm{T}} v_0} \cdot v_0) \in W$ and $\frac{\bar{v}^{\mathrm{T}} v_0}{\bar{v}_0^{\mathrm{T}} v_0} \cdot v_0 \in \mathbb{C}v_0$

Finally, the intersection of W and $\mathbb{C}v_0$ is zero because $\bar{v}_0^{\mathrm{T}} v_0$ is not zero. □

The Claim has been established. Thus, as was noted above the claim, the Theorem is also established.

We return to the regularly scheduled programming:

Theorem 2.16. (Cayley) *Every group is isomorphic to a group of permutations.*

Proof. Let G be a group. If $g \in G$, then let $g_L : G \to G$ be the function $g_L(g_1) = gg_1$ for all $g_1 \in G$. Notice that g_L is a permutation of G!

Let $G_L = \{g_L : G \to G \mid g \in G\}$.

Observe $(G_L, \circ)$ is a group.

- If $h, g \in G$, then $h_L \circ g_L = (hg)_L$. (So G_L is closed under $\circ$.)
- If e is the identity element of G, then e_L is the identity element of G_L.
- If $g \in G$, then $(g^{-1})_L = (g_L)^{-1}$.
- Function composition always associates.

Observe that $\phi : G \to G_L$, which is defined by $\phi(g) = g_L$ is a group isomorphism.

Indeed,

- we already saw that $\phi(hg) = \phi(h) \circ \phi(g)$,
- if g_L is an arbitrary element of G_L, for some $g \in G$, then $g_L = \phi(g)$,
- if $\phi(g) = \phi(h)$, then the functions g_L and h_L of G_L are equal. In particular, if e is the identity element of G, then

$$g = ge = g_L(e) = h_L(e) = he = h.$$

□

Corollary 2.17. *If G is a group of order[10] n, then G is isomorphic to a subgroup of S_n.*

Of course, there is more nothing to prove. The proof we gave also establishes the Corollary.

[10] The order of a group is the number of elements in the group.

Example 2.18. Lets use Cayley's Theorem to exhibit S_3 as a subgroup of S_6. Write S_3 as

$$a_1 = (1), \quad a_2 = (1,2), \quad a_3 = (1,3), \quad a_4 = (2,3), \quad a_5 = (1,2,3), \quad \text{and} \quad a_6 = (1,3,2).$$

Observe that

$$(a_2)_L : S_3 \to S_3$$

is

$$\begin{aligned} a_1 &\mapsto a_2 \\ a_2 &\mapsto a_1 \\ a_3 &\mapsto (1,2)(1,3) = (1,3,2) = a_6 \\ a_4 &\mapsto (1,2)(2,3) = (1,2,3) = a_5 \\ a_5 &\mapsto (1,2)(1,2,3) = (2,3) = a_4 \\ a_6 &\mapsto (1,2)(1,3,2) = (1,3) = a_3. \end{aligned}$$

So $(1,2)_L = (1,2)(3,6)(4,5)$. Similarly,

$$(a_5)_L : S_3 \to S_3$$

is

$$\begin{aligned} a_1 &\mapsto a_5 \\ a_2 &\mapsto (1,2,3)(1,2) = (1,3) = a_3 \\ a_3 &\mapsto (1,2,3)(1,3) = (2,3) = a_4 \\ a_4 &\mapsto (1,2,3)(2,3) = (1,2) = a_2 \\ a_5 &\mapsto (1,2,3)(1,2,3) = (1,3,2) = a_6 \\ a_6 &\mapsto (1,2,3)(1,3,2) = (1) = a_1. \end{aligned}$$

So $(1,2,3)_L = (1,5,6)(2,3,4)$.

In Homework 1, you saw that S_3 is generated by $(1,2)$ and $(1,2,3)$. Thus, the proof of Cayley's theorem shows that S_3 is isomorphic to the subgroup of S_6 which is generated by $(1,2)(3,6)(4,5)$ and $(1,5,6)(2,3,4)$.

Example 2.19. This is a more interesting example. Does there exist an 8-element group

$$\{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$$

which satisfies

$$a^4 = 1, \quad a^2 = b^2, \quad \text{and} \quad ba = a^3b?$$

Step 1. If there exists such a group; it could only have one multiplication table

	1	a	a^2	a^3	b	ab	a^2b	a^3b
1	1	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	1	ab	a^2b	a^3b	b
a^2	a^2	a^3	1	a	a^2b	a^3b	b	ab
a^3	a^3	1	a	a^2	a^3b	b	ab	a^2b
b	b	a^3b	a^2b	ab	a^2	a	1	a^3
ab	ab	b	a^3b	a^2b	a^3	a^2	a	1
a^2b	a^2b	ab	b	a^3b	1	a^3	a^2	a
a^3b	a^3b	a^2b	ab	b	a	1	a^3	a^2

We still do not know if this multiplication associates and we certainly do not know if all eight names are distinct. In Homework problem 7 in problem set three, I will ask you to apply the technique of the proof of Cayley's Theorem in order to identify an eight element subgroup of S_8 that has this multiplication table.

At home you obtained an eight element subgroup of S_8 with distinct elements of the form $a^i b^j$, with $0 \le i \le 3$ and $0 \le j \le 1$ whose elements satisfy $a^4 = 1$, $b^2 = a^2$, $ba = a^3 b$.

Remark. This problem is part of a larger problem; namely, find all groups of order 8. It turns out that (up to isomorphism) there are 5 groups of order 8: 3 Abelian groups, D_4, and this group. This group is called the Quaternion group Q_8. By the way, it is clear that D_4 and Q_8 are not isomorphic. Indeed, D_4 has 5 elements of order[11] 2 and 2 elements of order 4; whereas, Q_8 has 1 element of order 2 and 6 elements of order 4.

Actually, something much deeper is going on. It is reasonable to ask: suppose I have a group with a finite set of generators and and a finite set of relations. Is there an algorithm for determining if a given word is the identity element? This problem is called the "word problem". It was shown by Pyotr Novikov (1955) and William Boone (1958) that the word problem is undecidable. (Look at the Wikipedia page for the Word problem for groups.)

[11] The order of an element g in the group G is the number of elements in the subgroup of G which is generated by g. In particular, the order of g is the least positive integer n with g^n equal to the identity element, if such an integer n exists.

September 20, 2023

2.D. **Cyclic groups.**

Definition 2.20. The group G is a cyclic group if there exists an element g in G with

$$G = \{g^k \mid k \in \mathbb{Z}\}.$$

Examples 2.21. The group $(\mathbb{Z}, +)$ is cyclic of infinite order. The group $U_n = \{e^{\frac{2j\pi i}{n}} | j \in \mathbb{Z}\}$, which is equal to the group of n^{th} roots of 1 in $\mathbb{C}^*$, and the subgroup $\{(1, 2, \dots, n)^j | j \in \mathbb{Z}\}$ of S_n are cyclic groups of order n.

Observation 2.22. *Two cyclic groups are isomorphic if and only if they have the same order.*

Proof.
$(\Rightarrow)$ This direction is clear. An isomorphism is always a bijection.

$(\Leftarrow)$ We treat two cases: infinite cyclic groups and finite cyclic groups.

• We show that every infinite cyclic group is isomorphic to $(\mathbb{Z}, +)$. (This is good enough because the relation "are isomorphic" is an equivalence relation on the set of groups. You should prove this, if necessary.[12]) If G is a cyclic group with generator g and operation $*$, then

$$\phi : \mathbb{Z} \to G,$$

given by $\phi(j) = g^j$, is an isomorphism. Of course,

$$g^j \text{ means } \begin{cases} \underbrace{g * g * \cdots * g}_{j \text{ times}}, & \text{if } 0 < j, \\ \text{identity element}, & \text{if } j = 0, \\ \underbrace{g^{-1} * g^{-1} * \cdots * g^{-1}}_{|j| \text{ times}}, & \text{if } 0 < j. \end{cases}$$

(Please check, if necessary, that $\phi(j + k) = \phi(j) * \phi(k)$, ϕ is one-to-one, and ϕ is onto.)

• Suppose $A = \langle a \rangle$ and $B = \langle b \rangle$ are both cyclic groups of order n, where n is a finite positive integer. The elements of A are $\{a^j | 0 \le j \le n - 1\}$ and the elements of B are $\{b^j | 0 \le j \le n - 1\}$, where a^0 is the identity element of A and b^0 is the identity element of B. It is clear that

$$\phi : A \to B,$$

given by $\phi(a^j) = b^j$, for $0 \le j \le n - 1$, is a bijection. We show that ϕ is a homomorphism. If $0 \le i, j \le n - 1$, then $i + j = k + rn$ for some integers k and r with $0 \le k \le n - 1$. Observe that

$$\phi(a^i \cdot a^j) = \phi(a^{i+j}) = \phi(a^{k+rn}) = \phi(a^k \cdot (a^n)^r) = \phi(a^k) = b^k = b^{k+rn} = b^{i+j} = b^i \cdot b^j = \phi(a^i) \cdot \phi(a^j).$$

□

The next project is: What are the subgroups of a cyclic group?

[12] A relation $(\sim)$ on the set S is an equivalence relation if it is reflexive ($s \sim s$ for all $s \in S$), symmetric ($s \sim s' \Rightarrow s' \sim s$, for all $s, s' \in S$) and transitive ($s \sim s'$ and $s' \sim s''$ for $s, s', s'' \in S$ implies $s \sim s''$).

Example 2.23.

- What are the subgroups of $\mathbb{Z}$?
  Some subgroups that come to mind are:
  $$\langle 0\rangle, \quad \langle 1\rangle, \quad \langle 2\rangle, \quad \langle 3\rangle, \quad \langle 4\rangle, \quad \langle 5\rangle, \quad etc.$$
- What are the subgroups of U_{24}?
  Some subgroups that come to mind are
  $$U_1, \quad U_2, \quad U_3, \quad U_4, \quad U_6, \quad U_8, \quad U_{12}, \quad U_{24}.$$

Proposition 2.24. *Every subgroup of a cyclic group is cyclic.*

Proof. Let $G = \langle g\rangle$ be a cyclic and let H be a subgroup of G. If H consists only of the identity element, then H is certainly cyclic. Otherwise, there is some positive integer s with $g^s \in H$. Pick s to be the least positive integer with $g^s \in H$. We claim $H = \langle g^s\rangle$.

The inclusion $\supseteq$ is obvious.

We prove the inclusion $\subseteq$. Let $h = g^r$ be an arbitrary element of H. Write $r = \ell s + m$ for integers ℓ and m with $0 \le m \le s - 1$. It follows that $g^m \in H$. We picked s to have the property that if $1 \le i \le s - 1$, then $g^i \notin H$. Thus, $m = 0$ and

$$h = g^r = g^{\ell s+m} = g^{\ell s} = (g^s)^\ell \in \langle g^s\rangle.$$

□

Corollary 2.25. *If G is a finite cyclic group of order n, then G has exactly one subgroup of order d for each divisor d of n.*

Remark 2.26. In Example 2.23 we listed all of the subgroups of U_{24}.

Proof. Let $G = \langle g\rangle$. Fix a divisor d of n. Observe that $\langle g^{n/d}\rangle$ has order d. On the other hand, if H is a subgroup of G of order d, then the proof of Proposition 2.24 shows that $H = \langle g^s\rangle$ where s is the smallest positive exponent with g^s in H. Furthermore, the proof of Proposition 2.24 shows that this s must divide n (otherwise, there is a smaller exponent with g to that exponent is in H) and $\frac{n}{s}$ is the order of H. □

2.E. **Lagrange's Theorem.**

Theorem 2.27. *If H is a subgroup of a finite group G, then the order of H divides the order of G.*

We prove Lagrange's theorem by partitioning G into a bunch of cosets. Each element of G is in exactly one coset. Each coset has the same number of elements as H has.

Before I define coset, I want to point out that the set of cosets of H are an interesting mathematical object. They continue to be interesting even if H and G are infinite.

2.27.1. The group G acts on the set of cosets of H in G. We use this action when we prove the Sylow Theorems.

2.27.2. If H is a "normal" subgroup of G, then the set of cosets of H in G is a new group.

We first prove Lagrange's Theorem.

Theorem. *If H is a subgroup of a finite group G, then the order of H divides the order of G.*

Definition 2.28. If H is a subgroup of the group G and $g \in G$, then

- $gH = \{gh | h \in H\}$ is a left coset of H in G and
- $Hg = \{hg | h \in H\}$ is a right coset of H in G.

The proof of Lagrange's Theorem. Let H be a subgroup of the group G. (Assertions (a) and (b) hold even if the group G is infinite.) We show:

(a) Every element g of G is in exactly one left coset of H in G.
(b) There is a one-to-one correspondence between the the elements of H and the elements of gH for each element g in G.

Once (a) and (b) are established, we apply this information in the case that G is finite to conclude that

the number of elements in G = (the number of left cosets of H in G)×(the number of elements in H).

Proof of (a). Clearly $g \in gH$. If $g \in g_1H$, for some $g_1 \in G$, then we will prove that $gH = g_1H$. Well $g = g_1h_1$ for some $h_1 \in H$.
We show $gH \subseteq g_1H$: If $h \in H$, then $gh = g_1h_1h \in g_1H$.
We show $g_1H \subseteq gH$: If $h \in H$, then $g_1h = gh_1^{-1}h \in gH$.

Proof of (b). Let g be in G. Observe that the function $f : H \to gH$, which is given by $f(h) = gh$, for $h \in H$, is a bijection.

The function f is injective: If $f(h) = f(h')$, for $h, h' \in H$, then $gh = gh'$. Multiply both sides of the equation on the left by g^{-1} to conclude $h = h'$.

The function f is surjective: A typical element in the target of f is equal to gh for some h in H. We see that $f(h)$ is equal to this typical element. □

Corollary 2.29. *If G is a finite Abelian group, then G is cyclic if and only if the order of G is equal to the exponent of G.*

Remarks. (a) The order of the group G is the number of elements in G. The exponent of the group G is the least power n for which g^n is equal to the identity element for all $g \in G$.
(b) One consequence of Lagrange's Theorem is that if G is any finite group then $g^{|G|}$ is the identity element of G for every element g in G. (I used $|G|$ for the order of G.) Thus, the exponent of G is finite (when G is a finite group) and is at most the order of G
(c) In Corollary 2.29, the direction ⇒ is obvious.
(d) Corollary 2.29 would be false, if the hypothesis "Abelian" were removed. The group S_3 has order 6 and exponent 6, but is not cyclic.
(e) Corollary 2.29 is an immediate consequence of the structure theorem of finite Abelian groups. (You may use the structure Theorem of finite Abelian group to decide if some claim makes

sense; but you are not allowed to use it to prove results until we prove it.) At any rate, here is the invariant factor form of the structure of Finite Abelian Groups:

If G is a finite Abelian group, then G is isomorphic to

$$C_{d_1} \oplus C_{d_2} \oplus \dots \oplus C_{d_r} \tag{2.29.1}$$

for some positive integers $d_1, \dots, d_r$ with $d_1|d_2| \cdots |d_r$, where C_i is the cyclic group of order i.

It is clear that the exponent of the group (2.29.1) is d_r and the order of G is $\prod d_i$. It follows that the exponent of (2.29.1) is equal to the order of (2.29.1) if and only if $r = 1$.

There is also an elementary divisor form of this structure theorem. The invariant factor form corresponds to the rational canonical form of a matrix. The elementary divisor form of the structure of finite Abelian groups corresponds to the Jordan canonical form of a matrix.

We prove Corollary 2.29. We use two Lemmas. The second Lemma (2.31) requires that we know every integer can be factored uniquely into irreducible elements. This is one of my favorite Theorems. (Every PID is a UFD.) We aren't scheduled to prove it until the Chapter on Ring Theory; nonetheless, I like it too much to just fake it here. So, I will prove that the ring $\mathbb{Z}$ is a UFD, but I will do it in a group theory context. The general argument is exactly the same. I might skip it when we get there.

Note. The next Lemma reminds me of one of my favorite ways of getting test questions. Every result has hypotheses and every hypothesis is there for a reason. I often ask for an example that shows that a given hypothesis is necessary. This is also healthy way to study mathematics even if one is not thinking about exams.

Lemma 2.30. *Let x and y be elements in the group G. Suppose x and y have each have finite order, $xy = yx$, and $\langle x\rangle \cap \langle y\rangle = \langle \mathrm{id}\rangle$. Then the order of xy is equal to the least common multiple[13] of the order of x and the order of y.*

Proof. It is clear that $(xy)^{\mathrm{lcm}\{o(x),o(y)\}} = \mathrm{id}$. It suffices to prove that $o(x)$ and $o(y)$ both divide $o(xy)$.

Let $r = o(xy)$. It follows that $x^r = y^{-r} \in \langle x\rangle \cap \langle y\rangle = \langle \mathrm{id}\rangle$. Thus $x^r = y^r = \mathrm{id}$. Thus $o(x)|r$, $o(y)|r$, and the proof is complete. □

Lemma 2.31. *If x is an element of the finite Abelian group G and the order of x is maximal among all orders of elements of G, then the order of x is equal to the exponent of G.*

The proof of Lemma 2.31 uses the following Theorem.

Definition 2.32. A non-zero non-unit[14] element r of $\mathbb{Z}$ is irreducible if the only proper subgroup of $\mathbb{Z}$ which contains r is $\langle r\rangle$.

[13] We write lcm for least common multiple. The lcm of two integers a and b is the least non-negative integer that a and b both divide.

[14] The units of $\mathbb{Z}$ are $+1$ and -1.

Theorem 2.33. *Every non-zero non-unit of $\mathbb{Z}$ is equal to a finite product of irreducible elements of $\mathbb{Z}$. Furthermore, this factorization into irreducible elements is unique in the sense that if*

$$\prod_{i=1}^{a} r_i = \prod_{j=1}^{b} s_j,$$

with r_i and s_j irreducible integers, then $a = b$, and, after renumbering $r_i = s_i$ for all i.

Lemma 2.34. *The subgroups of $\mathbb{Z}$ satisfy the Ascending Chain Condition* (ACC). *In other words, every ascending chain of subgroups of $\mathbb{Z}$ stabilizes, in the following sense: If*

$$H_1 \subseteq H_2 \subseteq H_3 \subseteq \dots$$

is a chain of subgroups of $\mathbb{Z}$, then there exists an index n with H_n equal to H_m for all m with $n \le m$.

Remark 2.35. The subgroups of $\mathbb{Z}$ do not satisfy the Descending Chain Condition (DCC). Indeed,

$$\mathbb{Z} \supsetneq 2\mathbb{Z} \supsetneq 2^2\mathbb{Z} \supsetneq 2^3\mathbb{Z} \supsetneq \cdots$$

is an infinite properly decreasing chain of subgroups of $\mathbb{Z}$.

Proof of Lemma 2.34. Observe that $\cup_i H_i$ is a subgroup of $\mathbb{Z}$. Every subgroup of $\mathbb{Z}$ is cyclic. Thus $\cup_i H_i = \langle h \rangle$ for some $h \in \mathbb{Z}$. Thus $h \in H_n$, for some n, and $H_n = H_m$ for all m with $n \le m$. □

Lemma 2.36. *If n is an non-zero, non-unit integer, then $n \in \langle r \rangle$ for some irreducible integer r.*

Proof. Suppose n is not in $\langle r \rangle$ for any irreducible integer r. Then n is not irreducible, hence $n = n_0 n_0'$ with neither n_0 nor n_0' a unit.

But n_0 is not irreducible (otherwise $n \in \langle n_0 \rangle$ and n_0 is irreducible); thus, $n_0 = n_1 n_1'$ with neither n_1 nor n_1' a unit.

But n_1 is not irreducible (otherwise $n \in \langle n_1 \rangle$ and n_1 is irreducible); thus, $n_1 = n_2 n_2'$ with neither n_2 nor n_2' a unit.

We have produced an infinite strictly increasing chain of subgroups of $\mathbb{Z}$:

$$\langle n \rangle \subsetneq \langle n_0 \rangle \subsetneq \langle n_1 \rangle \subsetneq \dots \quad .$$

This is a contradiction. □

Lemma 2.37. *If n is a non-zero non-unit element of $\mathbb{Z}$, then n is a finite product of irreducible elements of $\mathbb{Z}$.*

Proof. Apply Lemma 2.36 multiple times

$$n = r_1 n_1 \text{ with } r_1 \text{ an irreducible integer and } n_1 \text{ an integer}$$

If n_1 is not a unit, then

$$n_1 = r_2 n_2 \text{ with } r_2 \text{ an irreducible integer and } n_2 \text{ an integer}$$

If n_2 is not a unit, then

$$n_2 = r_3 n_3 \text{ with } r_3 \text{ an irreducible integer and } n_3 \text{ an integer.}$$

Observe that if n_ℓ is not a unit, then

$$\langle n\rangle \subsetneq \langle n_1\rangle \subsetneq \langle n_2\rangle \subsetneq \langle n_3\rangle \subsetneq \cdots \subsetneq \langle n_{\ell+1}\rangle$$

is a strictly increasing chain of subgroups of $\mathbb{Z}$. The subgroups of $\mathbb{Z}$ satisfy (ACC); so, for some ℓ n_ℓ, is a unit and

$$n = r_1 \cdots r_{\ell-1}(r_\ell n_\ell)$$

is a finite product of irreducible integers. □

September 27, 2023

Due Monday Oct 2, HW3

Due Monday Oct 9, HW4

Exam Wed Oct 18

Are there questions?

Last time we proved that every integer (other than $0, 1, -1$) is equal to a finite product of irreducible integers.

We next prove that is factorization is unique in the sense that if

$$\prod_{i=1}^{s} p_i = \prod_{j=1}^{t} q_j,$$

with p_i and q_j irreducible integers, then $s = t$ and after re-numbering $\langle p_i\rangle = \langle q_i\rangle$ for each i.

Then we prove

Corollary. *If G is a finite Abelian group, then G is cyclic if and only if the order of G is equal to the exponent of G.*

The direction (⇐) is obvious. I owe you (⇒).

We prove 2 Lemmas in order to prove the Corollary.

We have a cool consequence of the Corollary. (But we give a proof that is not yet complete.)

We get to work:

Definition 2.38. The non-zero non-unit integer r is a prime integer if whenever a and b are integers with $ab \in \langle r\rangle$, then $a \in \langle r\rangle$ or $b \in \langle r\rangle$.

Observation 2.39. *Let n be an integer. Then n is prime if and only if n is irreducible.*

Proof. In this argument, n is a non-zero non-unit element of $\mathbb{Z}$.

Assume n is prime integer. We show that n is irreducible. Suppose $\langle a\rangle$ is a proper subgroup of $\mathbb{Z}$ and $n \in \langle a\rangle$. Thus $n = ab$ for some integer b and a is not a unit. The hypothesis that n is prime ensures that $a \in \langle n\rangle$ or $b \in \langle n\rangle$. Observe that $b \notin \langle n\rangle$. Indeed, if $b \in \langle n\rangle$, then $b = b'n$, for some integer b', and $n = ab = ab'n$[15]. Thus, $1 = ab'$ which is absurd because a is not a unit.

[15] Remember that we are thinking about the Abelian group $(\mathbb{Z}, +)$. When we write $n(1 - ab') = 0$, we mean n added to itself $(1 - ab')$ times is zero. Every non-zero element of $(\mathbb{Z}, +)$ has infinite order. The integer n is not zero; hence the integer $1 - ab'$ must be zero.

Thus $a \in \langle n \rangle$ and $a = a'n$, for some integer a', and $n = ab = a'nb$. It follows that $1 = a'b$ and $\langle n \rangle = \langle a \rangle$. We have shown that n is an irreducible element of $\mathbb{Z}$.

Now suppose that n is an irreducible element of $\mathbb{Z}$. We show that n is a prime element of $\mathbb{Z}$. Let a and b be integers with $a \notin \langle n \rangle$ and $b \notin \langle n \rangle$. Apply the definition of irreducible element to see that the subgroups $\langle n, a \rangle$ and $\langle n, b \rangle$ both must equal $\mathbb{Z}$. Thus, there are integers c_1, c_2, d_1, d_2 with

$$1 = c_1 n + c_2 a \quad 1 = d_1 n + d_2 b.$$

Observe that $1 \in \langle n, ab \rangle$. Conclude that $ab \notin \langle n \rangle$. Hence n is a prime element of $\mathbb{Z}$. □

Proof of Theorem 2.33. It suffices to prove that the factorization into irreducible elements is unique. Suppose

$$\prod_{i=1}^{a} r_i = \prod_{j=1}^{b} s_j,$$

with r_i and s_j irreducible integers. The integer r_1 is prime and $\prod_{j=1}^{b} s_j \in \langle r_1 \rangle$; thus some $s_j \in \langle r_1 \rangle$. Renumber the s's, if necessary, to obtain $s_1 \in \langle r_1 \rangle$. The integer s_1 is irreducible; hence $\langle s_1 \rangle = \langle r_1 \rangle$. Thus $s_1 = \pm r_1$ and

$$\prod_{i=2}^{a} r_i = \prod_{j=2}^{b} s_j.$$

Iterate (or induct) to finish the proof. □

We proved the result about factorization in order to prove the following two Lemmas.

Lemma. 2.30 *Let x and y be elements in the group G. Suppose x and y have each have finite order, $xy = yx$, and $\langle x \rangle \cap \langle y \rangle = \langle \mathrm{id} \rangle$. Then the order of xy is equal to the least common multiple*[16] *of the order of x and the order of y.*

Proof. It is clear that $(xy)^{\mathrm{lcm}\{o(x),o(y)\}} = \mathrm{id}$. It suffices to prove that $o(x)$ and $o(y)$ both divide $o(xy)$.

Let $r = o(xy)$. It follows that $x^r = y^{-r} \in \langle x \rangle \cap \langle y \rangle = \langle \mathrm{id} \rangle$. Thus $x^r = y^r = \mathrm{id}$. Thus $o(x)|r$, $o(y)|r$, and the proof is complete. □

Remark. Let x and y be elements of a group. Suppose x and y commute and have relatively prime order. Then the order of xy is the order of x times the order of y

Lemma. 2.31 *If x is an element of the finite Abelian group G and the order of x is maximal among all orders of elements of G, then the order of x is equal to the exponent of G.*

Proof. Let y be an element of G. Suppose that

$$\begin{array}{ll}\text{the order of } x \text{ is} & p_1^{e_1} \cdots p_s^{e_s}\\ \text{the order of } y \text{ is} & p_1^{f_1} \cdots p_s^{f_s},\end{array}$$

[16] We write lcm for least common multiple. The lcm of two integers a and b is the least non-negative integer that a and b both divide.

where $p_1, \dots, p_s$ are distinct positive prime integers. It suffices to show that $f_i \le e_i$ for all i. We prove this by contradiction. Renumber the p_i, if necessary, and suppose $e_1 < f_1$. We will draw a contradiction.

Observe that the order of $x^{p_1^{e_1}}$ is $p_2^{e_2} \cdots p_s^{e_s}$ and the order of $y^{p_2^{f_2} \cdots p_s^{f_s}}$ is $p_1^{f_1}$. Apply Lemma 2.30 to see that the order of $x^{p_1^{e_1}} y^{p_2^{f_2} \cdots p_s^{f_s}}$ is $p_1^{f_1} p_2^{e_2} \cdots p_s^{e_s}$. Thus we have manufactured an element in G which has order greater than the order of x. This is a contradiction. □

We are now ready to prove

Corollary. 2.29 *If G is a finite Abelian group, then G is cyclic if and only if the order of G is equal to the exponent of G.*

Proof of Corollary 2.29. We need only prove that if the finite Abelian group G has the same order and exponent, then G is cyclic. Let x be an element of G of maximal order. Then

$$\begin{aligned} \text{the order of } x &= \text{the exponent of } G, && \text{by Lemma 2.31} \\ &= \text{the order of } G, && \text{by hypothesis.} \end{aligned}$$

Thus $G = \langle x \rangle$ and G is cyclic. □

October 2, 2023

Due today HW3

Due Monday HW4

Exam Wed Oct 18

Are there questions?

Why do I want you to prove things from scratch, when you already know a big theorem that proves the statement instantly?

A partial proof of a cool corolary.

The arithmetic of cycles.

Quotient groups, normal subgroups, the isomorphism theorems

Last time we proved

Corollary. 2.29 *If G is a finite Abelian group, then G is cyclic if and only if the order of G is equal to the exponent of G.*

Corollary 2.40. *If F is a field, $F^* = (F \setminus \{0\}, \times)$ and G is a finite subgroup of F^*, then G is a cyclic group.*

My "proof" uses the fact that the polynomial ring $F[x]$ is a Unique Factorization Domain. We will eventually prove this fact. We will not have a real proof of Corollary 2.40 until we prove that $F[x]$ is a UFD.

"*Proof*" Let r be the exponent of G. It follows that $g^r = 1$ for all $g \in G$. The fact that $F[x]$ is a UFD guarantees that $x^r - 1$ has at most r roots in F. Thus,

$$\begin{aligned} \text{the exponent of } G &\leq \text{the order of } G, && \text{by Lagrange's Theorem,} \\ &\leq \text{the exponent of } G. && \text{We just showed this.} \end{aligned}$$

The group G is a finite Abelian group whose order is equal to its exponent. Apply Corollary 2.29 to conclude that G is a cyclic group. □

Remarks.
- The cleanest statement of Corollary 2.40 is that the multiplicative group of a finite field is cyclic.
- Once we prove the theorem about the structure of finite Abelian groups and Gauss' Lemma (that $F[x]$ is a UFD), then Corollary 2.40 is fairly easy to prove.
- The question "Let F be a finite field and let $F^\times$ be the multiplicative group $F \setminus \{0\}$. Describe the structure of the finite Abelian group $F^\times$. Prove that your description is correct." appeared on the Qual.

2.F. **The arithmetic of cycles.** There are seven thoughts about S_n in this subsection.

(1) Every permutation in S_n is equal to a product of disjoint cycles.

Proof. Let σ be an element of S_n. Decompose $\{1, \dots, n\}$ into disjoint orbits under the action of σ. (If $k \in \{1, \dots, n\}$, then the orbit of k under σ is $\{\sigma^i(k) \mid i \in \mathbb{Z}\}$.) Observe that $\sigma|_{\text{any fixed orbit}}$ is a cycle. Observe that $\sigma = \prod_{\text{all orbits}} \sigma|_{\text{orbit}}$, and this is a product of cycles. □

(2) Disjoint cycles in S_n commute.

Proof. If the cycles $(u_1, \dots, u_a)$ and $(v_1, \dots, v_b)$ are disjoint cycles in S_n, then the functions

$$(u_1, \dots, u_a)(v_1, \dots, v_b) \quad \text{and} \quad (v_1, \dots, v_b)(u_1, \dots, u_a)$$

are equal. □

(3) The order of a k-cycle is k. If $\sigma_1, \dots, \sigma_\ell$ are disjoint cycles, then the order of $\sigma_1 \cdots \sigma_\ell$ is the least common multiple of the

$$\{\text{order of } \sigma_1, \text{ order of } \sigma_2, \dots, \text{ order of } \sigma_\ell\}.$$

(See Lemma 2.30.)

(4) Every permutation in S_n is equal to a product of transpositions[17].

Proof. Observe that

$$(1, 2, 3, \dots, r) = (1, r)(1, r-1) \cdots (1, 4)(1, 3)(1, 2).$$

□

(5) The notion of even and odd permutation makes sense.

Observation 2.41. *Suppose that permutation σ in S_n is a product of a transpositions and also is a product of b transpositions. We claim that a and b are both even or a and b are both odd.*

Proof. It suffices to show that $(-1)^a = (-1)^b$. Observe that S_n acts on $\mathbb{Z}[x_1, \dots, x_n]$ by $\sigma(x_i) = x_{\sigma(i)}$. Let $\Delta = \prod\limits_{i<j}(x_j - x_i)$.

Claim 2.42. *If (k, ℓ) in S_n, then $(k, \ell)\Delta = -\Delta$.*

Proof of claim. It does no harm to assume that $k < \ell$. Observe that

$$\Delta = \Big(\prod_{\substack{i<j \\ \{i,j\}\cap\{k,\ell\}=\emptyset}} (x_j - x_i) \Big) \Big(\prod_{i<k} (x_k - x_i)(x_\ell - x_i) \Big) \Big(\prod_{k<i<\ell} (x_i - x_k)(x_\ell - x_i) \Big) \Big(\prod_{\ell<i} (x_i - x_\ell)(x_i - x_k) \Big)(x_\ell - x_k).$$

$$(k, \ell)(\Delta) = \Big(\prod_{\substack{i<j \\ \{i,j\}\cap\{k,\ell\}=\emptyset}} (x_j - x_i) \Big) \Big(\prod_{i<k} (x_\ell - x_i)(x_k - x_i) \Big) \Big(\prod_{k<i<\ell} (x_i - x_\ell)(x_k - x_i) \Big) \Big(\prod_{\ell<i} (x_i - x_k)(x_i - x_\ell) \Big)(x_\ell - x_k).$$

The four factors inside $\Big(\ \Big)$ remain unchanged. The factor $(x_\ell - x_k)$ has changed to $(x_k - x_\ell) = -(x_\ell - x_k)$. The claim is established. □

The observation follows readily, because $\sigma(\Delta) = (-1)^a \Delta$ and $\sigma(\Delta) = (-1)^b \Delta$. The polynomial Δ in the domain $\mathbb{Z}[x_1, \dots, x_n]$ is not identically zero; hence $(-1)^a = (-1)^b$, as desired. □

[17] A transposition is a 2-cycle.

Definition 2.43. If the element σ of S_n is equal to the product of an even number of transpositions, then σ is called an even permutation and if σ is equal to the product of an odd number of transpositions, then σ is called an odd permutation.

(6) Define the Alternating group and calculate its order.

Definition 2.44. The alternating group A_n is the following subgroup of S_n:

$$A_n = \{\sigma \in S_n | \sigma \text{ is an even permutation}\}.$$

Observation 2.45. *If* $2 \le n$*, then* A_n *has order* $\frac{n!}{2}$.

Proof. All of the odd permutations of S_n are in the coset $(1,2)A_n$. Indeed, S_n is the disjoint union of the cosets $(1)A_n \cup (1,2)A_n$. We saw, when we proved Lagrange's Theorem that all cosets of A_n in S_n have the same number of elements. It follows that the order of A_n is $\frac{1}{2}$ the order of S_n. (Of course, S_n has $n!$ elements.) □

(7) Calculate $\sigma(a_1, \dots, a_r)\sigma^{-1}$ and observe that the Klein 4-group is a normal subgroup of S_4.

Observation 2.46. *If* σ *and* $(a_1, \dots, a_r)$ *are permutations in* S_n*, then*

$$\sigma(a_1, \dots, a_r)\sigma^{-1} = (\sigma(a_1), \dots, \sigma(a_r)).$$

Proof. Observe that $\sigma(a_1, \dots, a_r)\sigma^{-1}$ and $(\sigma(a_1), \dots, \sigma(a_r))$ are the exact same function. Each one sends $\sigma(a_i)$ to $\sigma(a_{i+1})$ for $1 \le i \le r-1$, $\sigma(a_r)$ to $\sigma(a_1)$, and leaves

$$\{1, \dots, n\} \setminus \{\sigma(a_1), \dots, \sigma(a_r)\}$$

completely alone. □

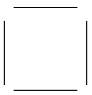
Example 2.47. The subgroup $V_4 = \{(1), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ of S_4 is closed under conjugation. In other words, if $\sigma \in S_4$ and $\tau \in V_4$, then $\sigma\tau\sigma^{-1}$ is in V_4. We learn in the next section that a subgroup which is closed under conjugation is called a normal subgroup. Felix Klein thought about this group V_4 and named it the Vierergruppe.

2.G. **Quotient groups, normal subgroups, the isomorphism theorems.**

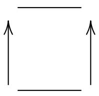
Make identifications to create new objects. Example 1. Surfaces.

One major technique that distinguishes Mathematics from many other disciplines is that in Mathematics one can take a perfectly good thing and one can pretend one part of the original thing is equal to some other part of the original thing and thereby create a brand new perfectly good thing.

The first example that comes to mind is the study of surfaces. One can start with a rectangular surface

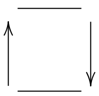


and pretend that each point on the left side is the same as the corresponding point in the right side.



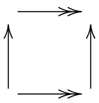
Now one has a cylinder.

Or one can start with a rectangular surface and pretend that each point on the left side is the same as the corresponding point in the right side measured in the opposite direction.



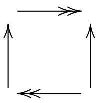
Now one has a Möbius bond.

One can make identifications on a rectangular surface



and create a torus.

One can make identifications on a rectangular surface



and create a Klein bottle.

The cylinder, the Möbius band, and the torus can all be built in 3-space. The Klein bottle can not be built in 3-space but it makes just as much sense to a Mathematician as the other three surfaces.

Oct. 4, 2023
HW4 is due on Monday.
HW5 will be posted soon. It is due on Monday, Oct. 16.
Exam 1 is Wednesday, Oct. 18.

Are there any questions?

Last time we took a topological space, made identifications, and produced a new topological space.

Make identifications to create new objects. Example 2. Groups.

Start with a group G. Pick out two elements g_1 and g_2. Our goal is to create a new group $\bar{G}$ which is as much like G as possible, but in which the image of g_1 in $\bar{G}$ is equal to the image of g_2 in $\bar{G}$. Lets write $\bar{g}_1$ in place of "the image of g_1 in $\bar{G}$".

Notice first that we are putting a relation $\sim$ on G and saying that $\bar{g}_1 = \bar{g}_2$ in $\bar{G}$ if and only if $g_1 \sim g_2$ in G. What kind of relations $\sim$ in G will give rise to groups $\bar{G}$?

(1) The relation $\sim$ better be an equivalence relation because $=$ in $\bar{G}$ is an equivalence relation. (See the footnote 12 on page 21 for the definition of an equivalence relation, if necessary.)
(2) If g_1, g_2, and g_3 are elements of G with $g_1 \sim g_2$, then one must have $g_1g_3 \sim g_2g_3$.
(3) In particular, $g_1 \sim g_2$ if and only if $g_1g_2^{-1} \sim e$ where e is the identity element of G.
(4) Hence, it suffices to figure out which elements g in G satisfy $g \sim e$. Let $N = \{g \in G \mid g \sim e\}$.
(5) Observe that N must be a subgroup.
(6) Observe that if $n \in N$ and g is an arbitrary element of G, then $gng^{-1} \sim geg^{-1} = e$; hence gng^{-1} must be in N.

Definition 2.48. If N is a subgroup of G and $gng^{-1} \in N$ for all $n \in N$ and $g \in G$, then G is called a normal subgroup of G.[18]

Remark. Sometimes it is easier to make sense of words than symbols. Here is Definition 2.48 expressed in words. A subgroup N of the group G is a normal subgroup if N is closed under conjugation by elements of G.

Definition 2.49. If N is a normal subgroup of G, then consider the set

$$\frac{G}{N} = \{\bar{g} \mid g \in G \text{ and } \bar{g}_1 = \bar{g}_2 \iff g_1g_2^{-1} \in N\}.$$

Remark. Here are two other ways two other ways to think of the set $\frac{G}{N}$.

- The set $\frac{G}{N}$ is the set of equivalence classes in G, where $g_1 \sim g_2$ if and only if $g_1g_2^{-1} \in N$.
- The set $\frac{G}{N}$ is the set of left cosets of N in G.

Theorem 2.50. *If N is a normal subgroup of the group G, then $\frac{G}{N}$ is a group with operation*

$$\bar{g}_1\bar{g}_2 = \overline{g_1g_2}.$$

Furthermore, the identity element of $\frac{G}{N}$ is $\bar{e}$, where e is the identity element of G and if g is an element of G, then the inverse of $\bar{g}$ is $\overline{g^{-1}}$.

Proof. **We must show that the proposed operation in $\frac{G}{N}$ makes sense.** In other words, suppose $\bar{g}_i = \bar{h}_i$ for $i \in \{1, 2\}$ and $g_1, g_2, h_1, h_2 \in G$. We must show that $\overline{g_1g_2} = \overline{h_1h_2}$.

Well, if $i \in \{1, 2\}$, then $g_i = h_in_i$ for some n_1, n_2 in N. Thus

$$g_1g_2 = h_1n_1h_2n_2 = h_1h_2(h_2^{-1}n_1h_2)n_2.$$

The subgroup N of G is normal so $h_2^{-1}n_1h_2 \in N$ and and $(h_2^{-1}n_1h_2)n_2 \in N$. Thus, $\overline{g_1g_2} = \overline{h_1h_2}$.

It is now completely trivial to show that $\frac{G}{N}$ satisfies all of the group axioms. □

Examples 2.51. (a) If G is an Abelian group, then every subgroup of G is normal.
(b) If n is a positive integer, then $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is the cyclic group of order n.

[18] The symbols "$N \lhd G$" mean "N is a normal subgroup of G".

(c) The subgroup $\langle(1,2)\rangle$ of S_3 is not normal because

$$(1,2,3)(1,2)(1,2,3)^{-1} = (2,3) \notin \langle(1,2)\rangle.$$

(d) The subgroup $\{(1),(1,2)(1,3),(1,3)(2,4),(1,4)(2,3)\}$ is a normal subgroup of S_4, A_4, and D_4. (See Example 2.47.)

(e) Every subgroup of index 2 is normal.

Proof. Let N be a subgroup of G of index two. Notice that if g is an element of G which is not in N, then

(2.51.1) G is the disjoint union of the left cosets $N \cup gN$.

We show that N is a normal subgroup of G. Take $n \in N$ and $g \in G$. If $g \in N$, then it is obvious that $gng^{-1} \in N$. Henceforth, $g \notin N$. We assume that $gng^{-1} \notin N$. We will reach a contradiction. If $gng^{-1} \notin N$, then by (2.51.1) $gng^{-1} \in gN$; hence $ng^{-1} \in N$ and $g^{-1} \in N$, which is impossible. □

(f) Consider the group Q_8, which is the eight element group generated by a, b with $a^4 = e$, $b^2 = a^2$, and $ba = a^3b$. The only element of Q_4 of order 2 is a^2. Observe that $\langle a^2\rangle$ is a normal subgroup of Q_8 (because conjugation preserves order. That is, if g, h are elements of a group, then g and hgh^{-1} have the same order.)

There is also another way to see that $\langle a^2\rangle \lhd Q_8$. If G is a group, then the center of G is

$$Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}.$$

It is true (and easy to see) that

$$Z(G) \lhd G$$

for all groups G. Furthermore, one can verify that $Z(Q_8) = \langle a^2\rangle$.

(g) Let $\phi : G \to G'$ be a group homomorphism and let

$$\ker\phi = \{g \in G \mid \phi(g) \text{ is equal to the identity element of } G'\}.$$

Then $\ker\phi$ is a normal subgroup of G.

Proof. Check that $\ker\phi$ is closed under the operation of G. Check that if $g \in \ker\phi$, then $g^{-1} \in \ker\phi$. Check that $\ker\phi$ is closed under conjugation. □

Theorem 2.52. [The First Isomorphism Theorem.] *Let $\phi : G \to G'$ be a group homomorphism.*

(a) *If N is a normal subgroup of G and $N \subseteq \ker\phi$, then ϕ induces a group homomorphism $\bar\phi : \frac{G}{N} \to G'$, with*

$$\bar\phi(\bar g) = \phi(g).$$

(b) *The homomorphism*

$$\bar\phi : \frac{G}{\ker\phi} \to \operatorname{im}\phi$$

is an isomorphism.

Remark 2.53. It is very difficult to produce homomorphisms from random groups. To create such a homomorphism, I usually view the random group as a quotient of a well-understood group, I create a homomorphism from the well understood group, and then I apply the First Isomorphism Theorem.

• Here is my first example of this philosophy. When we proved that all cyclic groups of order n are isomorphic, we gave an unpleasant argument. The "correct" argument is to show that any group of order n is isomorphic to $\frac{\mathbb{Z}}{n\mathbb{Z}}$:

Let G be a cyclic group of order n with generator g. (Call the operation in G "times".) Define $\phi : \mathbb{Z} \to G$ with $\phi(r) = g^r$. This is a homomorphism. Apply the First Isomorphism Theorem to conclude that $\bar{\phi} : \frac{\mathbb{Z}}{n\mathbb{Z}} \to G$ is an isomorphism.

• Here is a second example of this philosophy. The last time I taught the course, I put

Suppose that G is a group with 16 elements and $g^2 = \mathrm{id}$ for all $g \in G$, where id is the identity element of G.

(a) **Prove that G is Abelian.**

(b) **Prove that G is isomorphic to $C_2 \oplus C_2 \oplus C_2 \oplus C_2$, where C_2 is equal to the group of complex numbers $\{1, -1\}$ under multiplication.**

as one of the questions on the exam.

I was horrified how many students defined their isomorphism "the wrong way". The "right way" to define the isomorphism is **from $C_2 \oplus C_2 \oplus C_2 \oplus C_2$**.

Proof of the First Isomorphism Theorem, Theorem 2.52.

We must show that $\bar{\phi}$ of (a) **is a legitimate function.** Once we do that, then everything else is obvious.

Suppose that g_1 and g_2 are elements of G with $\bar{g}_1 = \bar{g}_2$ in $\frac{G}{N}$. We must show that $\phi(g_1) = \phi(g_2)$.

The hypothesis $\bar{g}_1 = \bar{g}_2$ in $\frac{G}{N}$ guarantees that

$$g_1 g_2^{-1} \in N \subseteq \ker \phi.$$

It follows that $\phi(g_1 g_2^{-1})$ is the identity element of G'; and therefore, $\phi(g_1) = \phi(g_2)$. □

Example 2.54. The groups $\frac{\mathbb{R}}{\mathbb{Z}}$ and U are isomorphic.

Proof. Consider the homomorphism $\phi : \mathbb{R} \to U$, which is given by $\phi(\theta) = e^{2\pi i\theta}$. Apply the First Isomorphism Theorem. □

Example 2.55. The groups $\frac{U}{U_2}$ and U are isomorphic.

Proof. Consider the homomorphism $\phi : U \to U$, which is given by $\phi(u) = u^2$. Apply the First Isomorphism Theorem. □

Example 2.56. The groups $\frac{U}{U_n}$ and U are isomorphic.

Proof. Consider the homomorphism $\phi : U \to U$, which is given by $\phi(u) = u^n$. Apply the First Isomorphism Theorem. □

Example 2.57. The groups $\frac{S_4}{V_4}$ and S_3 are isomorphic.

Proof. This one is sneaky. I do not know any homomorphisms from $S_4 \to S_3$. Instead, I propose that we consider $\phi : S_3 \to \frac{S_4}{V_4}$ to be the composition of the following two homomorphisms[19]:

$$S_3 \xrightarrow{\text{inclusion}} S_4 \xrightarrow{\text{natural quotient map}} \frac{S_4}{V_4}.$$

So, ϕ is automatically a homomorphism.

Observe that the kernel of ϕ is (1) because (1) is the only element of

$$V_4 \cap S_3.$$

Thus ϕ is an injection.[20]

An injective function from a six element set to a six element set is necessarily surjective. □

Example 2.58. The groups $\frac{S_n}{A_n}$ and U_2 are isomorphic.

Proof. Define $\phi : S_n \to U_2$ by

$$\phi(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Verify that ϕ is a homomorphism. Apply the First Isomorphism Theorem. □

Example 2.59. The groups $\frac{O_n(\mathbb{R})}{\mathrm{SO}_n(\mathbb{R})}$ and U_2 are isomorphic.

Proof. Define $\phi : O_n(\mathbb{R}) \to U_2$ by $\phi(M) = \det M$ for $M \in O_2(\mathbb{R})$. Apply the First Isomorphism Theorem. □

Example 2.60. If r and s are relatively prime integers[21], then

$$\frac{\mathbb{Z}}{rs\mathbb{Z}} \cong \frac{\mathbb{Z}}{r\mathbb{Z}} \oplus \frac{\mathbb{Z}}{s\mathbb{Z}}.$$

This assertion is usually called the Chinese Remainder Theorem.

Lemma 2.60.1. *If r and s are integers with greatest common divisor d[22], then there exist integers a and b with $ar + bs = d$.*

[19] If N is a normal subgroup of the group G, then the function $\phi : G \to \frac{G}{N}$, which is given by $\phi(g) = \bar{g}$, for all $g \in G$, is a group homomorphism. This homomorphism is called the natural quotient map.

[20] Have I ever said out loud that the homomorphism ϕ is an injection if and only if the kernel of ϕ consists of the identity element? At any rate, it is true, easy to prove, and very useful.

[21] We have established Theorem 2.33; so we have complete understanding of the phrase "relatively prime". In particular, "r and s are relatively prime" means that the only integers that divide both r and s are 1 and -1.

[22] Again, we have established Theorem 2.33 so we have complete understanding of the phrase "greatest common divisor". In particular, the greatest integer that divides both r and s is the greatest common divisor of r and s.

Proof. We proved in Proposition 2.24 that the smallest subgroup of $\mathbb{Z}$ that contains r and s, denoted $\langle r, s\rangle$, is cyclic. Let t be the name of the generator; so $\langle r, s\rangle = \langle t\rangle$. The integer $-t$ also generates $\langle r, s\rangle$. So change t to negative t, if necessary. We may assume that t is positive and $\langle r, s\rangle = \langle t\rangle$. The fact that $t \in \langle r, s\rangle$ ensures that $t = ar + bs$ for some integers a and b. We need only show that t is the greatest common divisor of a and b. The fact that $\langle r, s\rangle \subseteq \langle t\rangle$ ensures that t is a common factor of r and s. On the other hand, the equation $t = ar + bs$ guarantees that every common factor of r and s also divides t. □

Now prove the assertion of (2.60). Define $\phi : \mathbb{Z} \to \frac{\mathbb{Z}}{r\mathbb{Z}} \oplus \frac{\mathbb{Z}}{s\mathbb{Z}}$ by $\phi(n) = (\bar{n}, \bar{n})$. Observe that ϕ is a homomorphism.

We prove that ϕ is surjective. We know from Lemma 2.60.1 that there are integers a and b with $ra + bs = 1$. Observe that $\phi(1 - ar) = (\bar{1}, \bar{0})$ and $\phi(1 - bs) = (\bar{0}, \bar{1})$. Every element in the target can be written in terms of $(\bar{1}, \bar{0})$ and $(\bar{0}, \bar{1})$. We conclude that ϕ is surjective.

Observe that $\langle rs\rangle \subseteq \ker \phi$. Apply the first part of the First Isomorphism Theorem to conclude that

$$\bar{\phi} : \frac{\mathbb{Z}}{\langle rs\rangle} \to \frac{\mathbb{Z}}{r\mathbb{Z}} \oplus \frac{\mathbb{Z}}{s\mathbb{Z}}$$

is a group homomorphism. A surjective function from a set with rs elements to a set with rs elements is necessarily injective.

Example 2.61. Recall the group $D_n = \langle \sigma, \rho\rangle$, where σ is reflection across the x-axis and ρ is rotation (ccw) by $\frac{2\pi}{n}$. We know from Theorem 2.11.1 that D_n has $2n$ elements. We also know that $\sigma^2 = \text{id}$, $\rho^n = \text{id}$, and $(\rho\sigma)^2 = \text{id}$. How does one construct a homomorphism from D_n?

Theorem 2.61.1. *Let $\langle x, y\rangle$ be the free group on x and y and let N be the smallest*[23] *normal subgroup of $\langle x, y\rangle$ which contains x^2, y^n, and $(xy)^2$. Then the following statements hold.*

(a) *The groups D_n and $\frac{\langle x,y\rangle}{N}$ are isomorphic.*
(b) *If G is a group and g_1 and g_2 are elements of G with $g_1^2 = \text{id}$, $g_2^n = \text{id}$, and $(g_1g_2)^2 = \text{id}$, then there exists a group homomorphism $\Phi : D_n \to G$ with $\Phi(\sigma) = g_1$ and $\Phi(\rho) = g_2$.*

Proof. We first prove (a). Start with the homomorphism[24] $\phi : \langle x, y\rangle \to D_n$, given by $\phi(x) = \sigma$ and $\phi(y) = \rho$. Observe that x^2, y^n, and $(xy)^2$ are in $\ker \phi$ (which is a normal subgroup of $\langle x, y\rangle$). It follows that N (which is the smallest normal subgroup of $\langle x, y\rangle$ that contains x^2, y^n, and $(xy)^2$) is contained in $\ker \phi$. The First Isomorphism Theorem guarantees that there exists a homomorphism

$$\bar{\phi} : \frac{\langle x, y\rangle}{N} \to D_n,$$

with $\bar{\phi}(\bar{x}) = \sigma$ and $\bar{\phi}(\bar{y}) = \rho$. Observe that $\bar{\phi}$ is surjective and that the domain of $\bar{\phi}$ has at most $2n$ elements. Conclude that $\frac{\langle x,y\rangle}{N}$ has exactly $2n$ elements and $\bar{\phi}$ is an isomorphism.

[23] There does exist a smallest normal subgroup of $\langle x, y\rangle$ which contains x^2, y^2, and $(xy)^2$. Indeed, the set of normal subgroups of $\langle x, y\rangle$ which contains x^2, y^2, and $(xy)^2$ is not empty, because $\langle x, y\rangle$ is one such group. Thus, $N = \cap H$ as H roams over all normal subgroups of $\langle x, y\rangle$ which contains x^2, y^2, and $(xy)^2$.

[24] The group $\langle x, y\rangle$ is a free group, we are free to map the generators anywhere we want.

(b) The group $\langle x, y\rangle$ is a free group; so there exists a homomorphism $\psi : \langle x, y\rangle \to G$ with $\psi(x) = g_1$ and $\psi(y) = g_2$. Observe that $N \subseteq \ker \psi$. The First Isomorphism Theorem guarantees that there exists a homomorphism $\bar\psi : \frac{F}{N} \to G$ with $\bar\psi(\bar x) = g_1$ and $\bar\psi(\bar y) = g_2$. Let $\Phi : D_n \to G$ be the composition

$$D_n \xrightarrow{\bar\phi^{-1}} \frac{F}{N} \xrightarrow{\bar\psi} G.$$

Observe that $\Phi(\sigma) = g_1$ and $\Phi(\tau) = g_2$. □

Example 2.62. How does one construct a group homomorphism from the group Q_8? Recall that Q_8 is an eight element group with distinct elements of the form a^ib^j, with $0 \le i \le 3$ and $0 \le j \le 1$ whose elements satisfy $a^4 = 1$, $b^2 = a^2$, $ba = a^3b$.

Exercise 2.62.1. [25] *Let $\langle x, y\rangle$ be the free group on x and y and let N be the smallest normal subgroup of $\langle x, y\rangle$ which contains x^4, x^2y^{-2}, and $yxy^{-1}x^{-3}$. Then the following statements hold.*

(a) *The groups Q_8 and $\frac{\langle x,y\rangle}{N}$ are isomorphic.*

(b) *If G is a group and g_1 and g_2 are elements of G with $g_1^4 = \mathrm{id}$, $g_2^2 = g_1^2$, and $g_2g_1 = g_1^3g_2$, then there exists a group homomorphism $\Phi : Q_8 \to G$ with $\Phi(\sigma) = g_1$ and $\Phi(\rho) = g_2$.*

Theorem 2.63. [The Second Isomorphism Theorem.] *If K is a normal subgroup of the group G, then the following statements hold.*

(a) *There is a one-to-one correspondence between the subgroups of G which contain K and the subgroups of $\frac{G}{K}$. If H is a subgroup of G which contains K, then the corresponding subgroup of $\frac{G}{K}$ is $\frac{H}{K}$. If $\mathscr{H}$ is a subgroup of $\frac{G}{K}$, then the corresponding subgroup of G is*

$$\widehat{\mathscr{H}} = \{h \in G \mid \bar h \in \mathscr{H}\}.$$

(b) *If H is a subgroup of G with K a subgroup of H, then H is a normal subgroup of G if and only if $\frac{H}{K}$ is a normal subgroup of $\frac{G}{K}$.*

(c) *If H is a normal subgroup of G with K a subgroup of H, then*

$$\frac{\frac{G}{K}}{\frac{H}{K}} \cong \frac{G}{H}.$$

Proof. We prove (a).

- Let H be a subgroup of G containing K. Verify that $\frac{H}{K}$ is a subgroup of $\frac{G}{K}$.
- Let $\mathscr{H}$ be a subgroup of $\frac{G}{K}$. Verify that $\widehat{\mathscr{H}}$ is a subgroup of G which contains K.
- Let H be a subgroup of G containing K. verify that

$$\widehat{\frac{H}{K}} = H.$$

- Let $\mathscr{H}$ be a subgroup of $\frac{G}{K}$. Verify that

$$\frac{\widehat{\mathscr{H}}}{K} = \mathscr{H}.$$

[25] If the statement is true, then prove it. If the statement is false, then fix it and prove it.

We prove (b).

$H \lhd G \Rightarrow \frac{H}{K} \lhd \frac{G}{K}$:

If $\bar{h} \in \frac{H}{K}$ and $\bar{g} \in \frac{G}{K}$, then

$$\bar{g}\bar{h}\bar{g}^{-1} = \overline{ghg^{-1}} \in \frac{H}{K}.$$

$H \lhd G \Leftarrow \frac{H}{K} \lhd \frac{G}{K}$:

Take $h \in H$ and $g \in G$. Then $\bar{h} \in \frac{H}{K}$ and $\bar{g} \in \frac{G}{K}$. Thus, $\bar{g}\bar{h}\bar{g}^{-1} \in \frac{H}{K}$; but

$$\bar{g}\bar{h}\bar{g}^{-1} = \overline{ghg^{-1}}.$$

Thus $\overline{ghg^{-1}} \in \frac{H}{K}$ and $ghg^{-1} \in H$.

We prove (c). Consider the natural quotient map

$$G \xrightarrow{\theta} \frac{G}{H}.$$

Observe that $K \subseteq \ker\theta$. Apply the First Isomorphism Theorem to see that

$$\bar{\theta} : \frac{G}{K} \to \frac{G}{H},$$

given by $\bar{\theta}(gK) = gH$ is a well-defined group homomorphism. Apply the other part of the First Isomorphism Theorem to see that

$$\frac{\frac{G}{K}}{\ker\bar{\theta}} \cong \operatorname{im}\bar{\theta}.$$

Observe that $\bar{\theta}$ is surjective and $\ker\bar{\theta} = \frac{H}{K}$. Conclude that

$$\frac{\frac{G}{K}}{\frac{H}{K}} \cong \frac{G}{H}.$$

□

Example 2.64. What are the subgroups of $\frac{S_4}{V_4}$?

We know that the composition

$$S_3 \xrightarrow{inclusion} S_4 \xrightarrow{natural quotient map} \frac{S_4}{V_4}$$

is an isomorphism. We also know that the subgroups of S_3 are

$$\langle \mathrm{id} \rangle, \quad \langle (1,2) \rangle, \quad \langle (1,3) \rangle, \quad \langle (2,3) \rangle, \quad A_3, \quad \text{and} \quad S_3.$$

So the subgroups of $\frac{S_4}{V_4}$ are

$$\frac{V_4}{V_4}, \quad \frac{\langle (1,2) \rangle V_4}{V_4}, \quad \frac{\langle (1,3) \rangle V_4}{V_4}, \quad \frac{\langle (2,3) \rangle V_4}{V_4}, \quad \frac{A_3V_4}{V_4}, \quad \text{and} \quad \frac{S_3V_4}{V_4}.$$

Notice that if H is a subgroup of a group G and N is a normal subgroup of G, then

$$HN = \{hn \mid h \in H \text{ and } n \in N\}$$

is a subgroup of G and of course is the smallest subgroup of G which contains H and N. Observe that the above set HN is closed. If $h_1, h_2 \in H$ and $n_1, n_2 \in N$, then

$$(h_1n_1)(h_2n_2) = h_1h_2(h_2^{-1}n_1h_2)n_2$$

with h_1h_2 in H because H is a subgroup of G and $(h_2^{-1}n_1h_2)n_2$ in N because N is a normal subgroup of G. A different way to write the subgroups of S_4 which contain V_4 is

$$V_4, \quad \langle(1,2), V_4\rangle, \quad \langle(1,3), V_4\rangle, \quad \langle(2,3), V_4\rangle, \quad A_4, \quad \text{and} \quad S_4.$$

Theorem 2.65. [The Third Isomorphism Theorem.] *Let G be a group, N be a normal subgroup of G, and K be a subgroup of G. Then $K \cap N$ is a normal subgroup of K and*

$$\frac{KN}{N} \cong \frac{K}{K \cap N}.$$

Proof. There exists a homomorphism $\theta : K \to \frac{KN}{N}$, which is given by $\theta(k) = \bar{k}$. (This map is inclusion followed by the natural quotient map.) It is clear that θ is surjective and that the kernel of θ is $K \cap N$. □

2.H. **Groups acting on sets.**

Definition 2.66. The group G acts on the set S if there is a function

$$G \times S \to S,$$

written as

$$(g, s) \mapsto gs,$$

which satisfies:

(a) $\text{id}(s) = s$ for all $s \in S$, and
(b) $g(hs) = (gh)s$ for all $g, h \in G$ and $s \in S$.

Examples 2.67. (1) The group S_n acts on the set $\{1, 2, \dots, n\}$.
(2) The group $\text{GL}_n(\mathbb{R})$ acts on the set $\mathbb{R}^n$.
(3) Every group G acts on itself by left translation.
(4) Every group G acts on itself by conjugation.
(5) If H is a subgroup of G, then G acts on the set of left cosets of H in G by left translation.
(6) If $K \lhd G$, then G acts on K by conjugation.

Some Ideas 2.68. Let the group G act on the set S.

(1) If $x \in S$, then the orbit of x is $\{gx | g \in G\}$.
(2) The group G partitions the set S into a collection of disjoint orbits. For example, when $\text{SO}_2(\mathbb{R})$ acts on $\mathbb{R}^2$, then the action partitions xy-plane into the set of circles with center $(0, 0)$.
(3) If $x \in S$, then the stabilizer of x is $\text{stab}\, x = \{g \in G \mid gx = x\}$.

(4) Observe that if $x \in S$, then the orbit of x is equal to

$$\{gx | \text{where we take one representative from each left coset of } \operatorname{stab} x \text{ in } G\}.$$

Thus[26], $|\text{orbit } x| = [G : \operatorname{stab} x]$.

Conclusion 2.69. *If G is a group which acts on a finite set S, then*

$$|S| = \sum_x [G : \operatorname{stab} x],$$

where the sum is taken over one element x from each orbit.

Application 2.70. *Let G be a finite group and let G act on itself by conjugation. The orbits of this action are the set of conjugacy classes of G.*[27] *If $g \in G$, then*

$$\operatorname{stab} g = \{h \in G \mid hgh^{-1} = g\}.$$

This set is called the centralizer of g in G. One obtains the equation

$$|G| = \sum_{x_i} [G : C(x_i)],$$

where one x_i is taken from each conjugacy class of G. It is often useful to separate the conjugacy classes which have exactly one element.

Theorem 2.71. [The Class Equation] *If G is a finite group with center*[28] *C, then*

$$|G| = |C| + \sum_{x_i} [G : C(x_i)],$$

where one x_i is taken from each conjugacy class of G which contains more than element.

Corollary 2.72. *Let p be a prime integer and n be a positive integer. If G is a group of order p^n, then G has a non-trivial center.*

Remark. The assertion of Corollary 2.72 is that the center of G is larger than merely the identity element of G.

Proof. The Class equation gives that

$$|G| = |C| + \sum_{x_i} [G : C(x_i)],$$

where one x_i is taken from each conjugacy class of G which contains more than element. Observe that $1 \le |C|$, p divides $|G|$, and p divides each $[G : C(x_i)]$ that appears. Thus $1 < |C|$. □

Theorem. *If G is a group of order p^n for some n with $1 \le n$, then G has a non-trivial center.*

[26] I am writing $[G : H]$ for the number of left cosets of H in G. I used a slightly different notation in Homework problem 8. This number is called the index of H in G.

[27] If $g \in G$, then the conjugacy class of g in G is $\{hgh^{-1} \mid h \in G\}$.

[28] The center of the group G is the set of elements of G that commute with every element of G.

Proof. Let G act on G by conjugation. Then

$$\underbrace{|G|}_{p\text{ divides this}} = \overbrace{\underbrace{|\{g \in G | \{g\} = \text{orbit}(g)\}|}_{\text{This number is not zero}}}^{\text{the center of } G} + \sum_g \underbrace{|\text{orbit}(g)|}_{\underbrace{[G : \text{stab}\, g]}_{p\text{ divides this}}},$$

where the sum is taken over the orbits of size larger than one and exactly one g is taken from each orbit. □

Corollary 2.73. *If G is a group of order p^2, where p is a prime integer, then G is Abelian.*

Proof. Let C be the center of G. Apply Corollary 2.72 to see that $1 < |C|$. Apply Lagrange's Theorem to see that $|C|$ is equal to p or p^2.

It suffices to prove that $p \neq |C|$. Assume $|C| = p$. We will reach a contradiction. It is clear that $C \lhd G$. Thus, $|\frac{G}{C}| = p$. Apply Lagrange's Theorem to see that $\frac{G}{C}$ is a cyclic group. It follows that there is an element $g \in G$ such that every element of G has the form $g^i c$ for some i and some $c \in C$. It is clear that $g^i c$ and $g^j c'$ commute for all integers i and j and all elements c and c' in C. We have proven that G is an Abelian group and this is absurd because, the center of G is a proper subgroup of G. □

Corollary 2.74. *If p is a prime integer, then every group of order p^2 is isomorphic to $\frac{\mathbb{Z}}{p^2\mathbb{Z}}$ or $\frac{\mathbb{Z}}{p\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p\mathbb{Z}}$.*

Proof. Let G be a non-cyclic group of order p^2. By Lagrange's Theorem every non-identity element of G has order p. Let a be one of these elements. Take $b \in G \setminus \langle a \rangle$. Observe that $\langle b \rangle \cap \langle a \rangle = \{\text{id}\}$. Otherwise, there exists i with b^i equal to a non-identity element of $\langle a \rangle$. Every non-identity element of $\langle a \rangle$ generates $\langle a \rangle$. In this case,

$$\langle a \rangle \subsetneq \langle b \rangle$$

and each group has order p. This of course, is absurd.

Use the First Isomorphism Theorem to see that there are group homomorphisms $\frac{\mathbb{Z}}{p\mathbb{Z}} \to G$ given by

$$\bar{n} \mapsto a^n \quad \text{and} \quad \bar{m} \mapsto b^m.$$

We proved in Corollary 2.73 that G is Abelian, so we may apply the Universal Mapping Property for direct sum of Abelian groups to see that there exists a homomorphism

$$\phi : \frac{\mathbb{Z}}{p\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p\mathbb{Z}} \to G$$

with

$$\phi((\bar{n}, \bar{m})) = a^n b^m.$$

The image of ϕ is a subgroup of G of order greater than p (because $\langle a \rangle$ has order p and $b \notin \langle a \rangle$. The only subgroup of G which has order larger than p is G itself. Thus, ϕ is surjective. Every surjective function from a set of size p^2 to a set of size p^2 is necessarily injective. Thus, ϕ is an isomorphism. □

2.I. **The Sylow Theorems.**

Definition 2.75. Let G be a finite group and p be a prime integer which divides the order of G. If p^r divides the order of G and p^{r+1} does not divide the order of G, then any subgroup of G of order p^r is called a Sylow p-subgroup of G.

Theorem 2.76. *Let G be a finite group and p be a prime integer which divides the order of G. Then the following statements hold.*

(a) *If p^r divides the order of G, then G has a subgroup of order p^r.*
(b) *Every subgroup of G of order p^r is contained in a Sylow p-subgroup of G.*
(c) *Any two Sylow p-subgroups of G are conjugate.* (*In other words, if H_1 and H_2 are Sylow p-subgroups of G, then there is an element $g \in G$ such that $gH_1g^{-1} = H_2$.*)
(d) *If n is the number of Sylow p-subgroups then*
   (i) *n divides the index* $[G : \text{a Sylow } p\text{-subgroup}]$ *and*
   (ii) $n \equiv 1 \mod p$.

The First Step 2.77. (Cauchy's Theorem) *Let G be a finite group and p be a prime integer which divides the order of G. Then G has an element of order p.*

Warm Up. Think about $p = 2$. Pair every element up with its inverse. Some of these pairings have size 2; the pairing that goes with id has size 1. The group has even size; thus there must be some non-identity element which is its own inverse.

We want to generalize this approach to work for all p. I propose that we think of the set of tuples of length 2 so that the product of the two elements is the identity. Let $\mathbb{Z}/2\mathbb{Z}$ act on this set by cyclic permutation. The set of such tuples decomposes into disjoint orbits. We want to count the number of orbits which have size 1.

Proof. Let

$$S = \{(a_1, \dots, a_p) \mid a_i \in G \text{ and } a_1 \cdots a_p = \text{id}\}.$$

Observe that $|S| = |G|^{p-1}$. Indeed, one can pick $a_1, \dots, a_{p-1}$ at random and then one is forced to choose $a_p = (a_1 \cdots a_{p-1})^{-1}$. Let $\frac{\mathbb{Z}}{p\mathbb{Z}}$ act on S by cyclic permutation:

$$k(a_1, \dots a_p) = (a_{k+1}, \dots, a_p, a_1, \dots, a_k).$$

This is an action because

$$0(a_1, \dots a_p) = (a_1, \dots a_p)$$

and

$$k'(k((a_1, \dots a_p)) = (k' + k)(a_1, \dots a_p).$$

Thus,

$$|S| = |\{s \in S \mid |\text{orbit of s}| = 1\}| + \sum |\text{orbit of } s|,$$

where the sum is taken over one s from each orbit with at least two elements. We know that p divides $|S|$. If the orbit of s has more than one element, then

$$|\text{the orbit of s}| = [\tfrac{\mathbb{Z}}{p\mathbb{Z}} : \text{stab}\, s]$$

and this number is divisible by p. Thus, p divides $|\{s \in S \mid |\text{orbit of s}| = 1\}|$ and there is an element x in G, with x not the identity element and $x^p = \text{id}$. □

The Next Step 2.78. *We prove assertions* (a) *and* (b) *of Theorem* 2.76.

Let G be a finite group. Suppose p is a prime integer and p divides the order of G. The following statements hold.

(a) If p^n divides the order of G, then G has a subgroup of order p^n.
(b) Every subgroup of G of order p^n is contained in some Sylow p-subgroup of G.

Proof. The proof is by induction. Suppose p^r divides the order of G and p^{r+1} does not divide the order of G. Suppose that H is a subgroup of G of order p^n for some n with $1 \le n \le r-1$. We prove that there exists a subgroup H_1 of G with $H \subseteq H_1$ and $|H_1| = p^{n+1}$.

Let H act on the left cosets of H in G by left translation.

Let S be the set of left cosets of H in G. We see that

$$|S| = |\{s \in S| \text{ the orbit of } s \text{ has one element}\}| + \sum |\text{the orbit of } s|,$$

where the sum is taken over one s from each large orbit.

Observe that $|S| = [G : H]$. The hypothesis ensures that p divides this number.

If the orbit of s has more than one element, then $|\text{orbit of } s| = [H : \operatorname{stab} s]$. We arranged that $[H : \operatorname{stab} s] \neq 1$. Thus p divides $[H : \operatorname{stab} s]$. It follows that

$$p \text{ divides } |\{s \in S| \text{ the orbit of } s \text{ has one element}\}|.$$

Thus,

$$p \text{ divides } |\{xH|hxH = xH \text{ for all } h \in H\}| \textit{ and}$$

$$p \text{ divides } |\{xH|x^{-1}hx \in H \text{ for all } h \in H\}|.$$

Observe that $\{x \in G|x^{-1}hx \in H \text{ for all } h \in H\}$ is a subgroup of G. This subgroup is called the normalizer of H in G. It might be denoted as $N(H)$ or $N_G(H)$. At any rate $H \lhd N(H)$. Thus,

$$\frac{N(H)}{H}$$

is a legitimate group and we have shown that p divides the order of this group. Apply 2.77 to see that

$$\frac{N(H)}{H}$$

has an element of order p. In other words, $N(H)$ has a subgroup of order p^{n+1} and this subgroup contains H. □

The Next Step 2.79. *We prove assertion* (c) *of Theorem* 2.76.

Proof. Let P and H both be Sylow subgroups of G.

Let H act on the set of left cosets of P in G.

Then

$$|S| = |\{s|\text{orbit of } s \text{ has size } 1\}| + \sum_s |\text{the orbit of } s|,$$

where the sum includes exactly one s from each orbit of size more than one.

Of course S is the set of left cosets of P in G; hence $|S| = [G : P]$ and p does not divide this number. If the orbit of s has more than one element, then

$$|\text{the orbit of } s| = [H : \operatorname{stab} s]$$

and p does divide this number. Thus, p does not divide

$$|\{s|\text{orbit of } s \text{ has size } 1\}|.$$

In particular,

$$|\{s|\text{orbit of } s \text{ has size } 1\}| \neq 0.$$

Hence, there is a left coset xP of P in G with the property that $hxP = xP$ for all $h \in H$. Thus, $x^{-1}hx \in P$ for all h in H for some $x \in G$. Thus, $x^{-1}Hx \subseteq P$. Both sets have the same size. We conclude that $x^{-1}Hx = P$. □

The Next Step 2.80. *We prove assertion* (di) *of Theorem* 2.76.

Proof. **Let G act on the set of Sylow p-subgroups of G by conjugation.** In light of (c) there is only one orbit; thus,

$$\text{the number of Sylow } p\text{-subgroups of } G = [G : \operatorname{stab} P],$$

where P is any fixed Sylow p-subgroups of G. Observe that

$$\operatorname{stab} P = \{x \in G \mid xPx^{-1} = P\} = N(P).$$

Thus,

$$\text{the number of Sylow } p\text{-subgroups of } G = [G : N(P)],$$

and this number divides $[G : P]$ because

$$[G : P] = [G : N(P)][N(P) : P].$$

See HW8. □

The Next Step 2.81. *We prove assertion* (dii) *of Theorem* 2.76.

Proof. Let P be a Sylow p-subgroup of G.

Let P act on the set of Sylow p-subgroups of G by conjugation.

Obtain

$$\text{the number of Sylow } p\text{-subgroups of } G = |\{s|\text{the orbit of } s \text{ has one element}\}| + \sum_s |\text{the orbit of } s|$$

where the sum is taken over the set of orbits of size more than 1 and exactly one s is taken from each such orbit.

If the orbit of s has size more than 1 then the orbit of s has $[P : \operatorname{stab} s]$ elements. This number is divisible by p.

It is clear that P is an element of S with orbit size 1. We complete the proof by showing that P is the only Sylow p-subgroup with orbit size 1.

Suppose that Q has orbit size 1. Then $xQx^{-1} = Q$ for all $x \in P$. Thus, $P \subseteq N(Q)$. The groups P and Q are both Sylow p-subgroups of $N(Q)$. According to (c), P and Q are conjugate in $N(Q)$. On the other hand Q is a normal subgroup of $N(Q)$. So, $P = gQg^{-1} = Q$ for some $g \in N(Q)$. The equality on the left holds because P and Q are conjugate in $N(Q)$; the equality on the right holds because, $Q \lhd N(Q)$. □

2.I.a. *First Application of the Sylow Theorems.*

Observation 2.82. *If G is a group of order pq, where $q < p$ are prime integers and q does not divide $p - 1$, then G is cyclic.*

Example. Every group of order 15 is cyclic.

Proof. We show

(1) G has an element a of order p and an element b of order q;
(2) the subgroups $\langle a\rangle$ and $\langle b\rangle$ are normal subgroups of G;
(3) $\langle a\rangle \cap \langle b\rangle = \{\mathrm{id}\}$;
(4) $ab = ba$;
(5) ab has order pq.

(1) This assertion is an immediate application of the Sylow Theorems.

(2) The number of Sylow p-subgroups of G (denoted n_p) is congruent to 1 mod p and divides q. Thus $n_p = 1$. The number of Sylow q-subgroups of G (denoted n_q) is congruent to 1 mod q and divides p. (If $(aq + 1)|p$, with a positive, then $aq + 1 = p$ and $aq = p - 1$, which has been ruled out by hypothesis.) Thus, n_q is also 1. It follows that $\langle a\rangle$ and $\langle b\rangle$ both are normal subgroups of G.

(3) Every non-identity element in $\langle a\rangle \cap \langle b\rangle$ has order p and also has order q. Of course, this makes no sense. Thus, $\langle a\rangle \cap \langle b\rangle = \{\mathrm{id}\}$.

(4) Observe that

$$(aba^{-1})b^{-1} = a(ba^{-1}b^{-1}) \in \langle a\rangle \cap \langle b\rangle = \{\mathrm{id}\},$$

because $\langle a\rangle$ and $\langle b\rangle$ both are normal subgroups of G. Thus, $ab = ba$.

(5) Apply Lemma 2.30 or just notice that ab does not have order 1, p, or q. □

2.I.b. *Second Application of the Sylow Theorems.* In this section we classify the non-Abelian groups of order 12. (We classify all finite Abelian groups in the next section. There is no need to do another special case of that classification here.) We use Lemma 2.83 in our classification. The easiest way to describe one of the groups of order 12 is by using "semidirect product". This technique is introduced in Observation 2.84. You might find Keith Conrad's notes [3] about this classification to be interesting.

Lemma 2.83. *The only subgroup of S_n of index two is A_n.*

Proof. [29] Let H be a subgroup of S_n of index two. Thus, H is a normal subgroup of S_n and S_n/H is isomorphic to U_2. Let $\phi : S_n \to U_2$ be a surjective homomorphism with kernel H. Observe that all transpositions in S_n are conjugate. Thus, ϕ carries every transposition of S_n to the same value in the Abelian group U_2. The transpositions generate S_n; hence $\phi(\sigma)$ generates U_2 as σ roams over the transpositions of S_n. It follows that $\phi(\sigma) = -1$ for each transposition σ of S_n and the kernel of S_n is necessarily equal to A_n. We have shown $H = \ker \phi = A_n$. □

Recall that if N is a group, then $\mathrm{Aut}(N)$ is the set of group isomorphisms $N \to N$. Recall also that $\mathrm{Aut}(N)$ is a group in its own right with the operation composition. Let N and H be groups and $\phi : H \to \mathrm{Aut}(N)$ be a group homomorphism. We form a new group $N \rtimes_\phi H$, called the semidirect product of N and H. The elements of $N \rtimes_\phi H$ are

$$\{(n,h) \mid n \in N \text{ and } h \in H\}$$

The operation is

$$(n_1,h_1)\cdot(n_2,h_2) = (n_1\phi(h_1)|_{n_2}, h_1h_2).$$

Observation 2.84. *If N and H are groups and $\phi : H \to \mathrm{Aut}(N)$ is a group homomorphism, then $N \rtimes_\phi H$ is a group. The identity element of $N \rtimes_\phi H$ is $(\mathrm{id}_N, \mathrm{id}_H)$. The inverse of (n,h) is $(\phi(h^{-1})|_{n^{-1}}, h^{-1})$. The set $\{(n, \mathrm{id}_H) \mid n \in N\}$ is a normal subgroup of $N \rtimes_\phi H$.*

Proof. **identity element**

$$(n,h)\cdot(\mathrm{id}_N,\mathrm{id}_H) = (n\phi(h)|_{\mathrm{id}_N}, h\,\mathrm{id}_H)$$

Every homomorphism (in particular $\phi(h)$) carries the identity element (in particular id_N in N) to the identity element (in this case id_N in N).

$$= (n,h).$$

$$(\mathrm{id}_N,\mathrm{id}_H)\cdot(n,h) = (\mathrm{id}_N\,\phi(\mathrm{id}_H)|_n, \mathrm{id}_H\,h)$$

Every homomorphism (in this case ϕ) carries the identity element (in this case, id_H in H) to the identity element (in this case, the Automorphism of N which sends each element to itself).

$$= (\mathrm{id}_N\,n, \mathrm{id}_H\,h) = (n,h).$$

inverse

$$(n,h)\cdot(\phi(h^{-1})|_{n^{-1}}, h^{-1}) = (n \underbrace{\phi(h)|_{\phi(h^{-1})|_{n^{-1}}}}_{\left(\underbrace{\phi(h)\circ\phi(h^{-1})}_{\phi(hh^{-1})}\right)\Big|_{n^{-1}}}, hh^{-1}) = (nn^{-1}, hh^{-1}) = (\mathrm{id}_N, \mathrm{id}_H)$$

[29] I found this proof at https://math.stackexchange.com/questions/27024/a-n-is-the-only-subgroup-of-s-n-of-index-2

$$(\phi(h^{-1})|_{n^{-1}}, h^{-1}) \cdot (n,h) = (\underbrace{\phi(h^{-1})|_{n^{-1}}\phi(h^{-1})|_{n}}_{\phi(h^{-1})|_{n^{-1}n}}, h^{-1}h) = (\phi(h^{-1})|_{\mathrm{id}_N}, \mathrm{id}_H) = (\mathrm{id}_N, \mathrm{id}_H).$$

The assertion that the inverse of the inverse of (n,h) is (n,h) is true and the proof is interesting.

associativity On the one hand,

$$\begin{aligned}
&\Big((n_1,h_1)\cdot(n_2,h_2)\Big)\cdot(n_3,h_3)\\
=&\Big(n_1\phi(h_1)|_{n_2}, h_1h_2\Big)\cdot(n_3,h_3)\\
=&\Big(n_1\phi(h_1)|_{n_2}\phi(h_1h_2)|_{n_3}, (h_1h_2)h_3\Big).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(n_1,h_1)\cdot\Big((n_2,h_2)\cdot(n_3,h_3)\Big)\\
=&(n_1,h_1)\cdot\Big((n_2\phi(h_2)|_{n_3}, h_2h_3)\Big)\\
=&(n_1\phi(h_1)|_{(n_2\phi(h_2)|_{n_3}}, h_1(h_2h_3)).
\end{aligned}$$

These are equal.

The set $\{(n,\mathrm{id}_H) \mid n\in N\}$ is a normal subgroup of $N \rtimes_\phi H$.

Let (n_1,h_1) be an arbitrary element of $N\rtimes_\phi H$. Observe that

$$\begin{aligned}
&(\phi(h_1^{-1})|_{n_1^{-1}}, h_1^{-1})(n,\mathrm{id})(n_1,h_1)\\
=&(\phi(h_1^{-1})|_{n_1^{-1}}\phi(h_1^{-1})|_{n}, h_1^{-1})(n_1,h_1)\\
=&(\text{an element of } N, h_1^{-1}h_1)\checkmark
\end{aligned}$$

□

Example 2.85. If H and N are subgroups of a group G with N a normal subgroup of G and $NH = G$, then define $\phi : H \to \operatorname{Aut} N$ by $\phi(h)$ is the homomorphism $\phi(h) : N \to N$ which sends n to $\phi(h)|_n = hnh^{-1}$. Observe that G is isomorphic to $N \rtimes_\phi H$. The details are left to you.

Example 2.86. If H and N are groups and $\phi : H \to \operatorname{Aut} N$ is the homomorphism $\phi(h)$ is the identity function $N \to N$ for all h in H. Then $N\rtimes_\phi H$ is the direct product $N\times H$.

Example 2.87. Let $\phi : \mathbb{Z}/4\mathbb{Z} \to \operatorname{Aut}(\mathbb{Z}/3\mathbb{Z})$ be the homomorphism with

$$\phi(\bar{b})|_{\bar{c}} = (-1)^b\bar{c},$$

for all $\bar{b} \in \mathbb{Z}/4\mathbb{Z}$ and $\bar{c}\in\mathbb{Z}/3\mathbb{Z}$. (We say this a little more slowly: ϕ is a homomorphism from $\mathbb{Z}/4\mathbb{Z}$ to $\operatorname{Aut}(\mathbb{Z}/3\mathbb{Z})$. If $\bar{b}$ is in $\mathbb{Z}/4\mathbb{Z}$, then $\phi(\bar{b})$ is an automorphism of $\mathbb{Z}/3\mathbb{Z}$. If $\bar{b}$ is in $\mathbb{Z}/4\mathbb{Z}$ and $\bar{c}\in\mathbb{Z}/3\mathbb{Z}$, then $\phi(\bar{b})$ sends $\bar{c}$ to $(-1)^b\bar{c}$.)[30] The group $(\mathbb{Z}/3\mathbb{Z})\rtimes_\phi(\mathbb{Z}/4\mathbb{Z})$ is called the dicyclic group. Then

(1) the dicyclic group has 12 elements,

[30] If n and a are integers, we write $\bar{a}$ for the class of a in $\mathbb{Z}/n\mathbb{Z}$.

(2)
- the dicyclic group has 2 elements of order 6,
- the dicyclic group has 6 elements of order 4,
- the dicyclic group has 2 elements of order 3,
- the dicyclic group has 1 element of order 2, and
- the dicyclic group has 1 element of order 1, and

(3) there are elements x, y in the dicyclic group such that the dicyclic group is equal to $\langle x, y\rangle$, $x^6 = \text{id}$, $y^2 = x^3$, $yxy^{-1} = x^5$.

(4) Furthermore, if F is the free group on X, Y, and N is the smallest normal subgroup of F which contains X^6, Y^2X^4, and $YXY^{-1}X$, then F/N is isomorphic to the dicyclic group.

(5) If G is a group with 12 elements $G = \langle \xi, \psi\rangle$, $\xi^6 = \text{id}$, $\psi^2 = \xi^3$, and $\psi\xi\psi^{-1} = \xi^{-1}$, then G is isomorphic to the dicyclic group.

You will establish most of these assertions for homework.

Theorem 2.88. *If G is a non-Abelian group of order* 12*, then G is isomorphic to exactly one of the following groups:*

$$A_4, \quad D_6, \quad \textit{or} \quad \textit{the dicyclic group.}$$

Proof. No two of the three listed groups are isomorphic:

- A_4 has 3 elements of order 2 and 8 elements of order 3;
- D_6 has 7 elements of order 2, 2 elements of order 3, and 2 elements of order 6;
- and the dicyclic group has 1 element of order 2, 2 elements of order 3, 6 elements of order 4, and 2 elements of order 6.

Observe that G has at least one Sylow 3-subgroup; call it P. Let G act on the set of cosets of P in G by left translation. This action is equivalent to a group homomorphism $\phi : G \to S_4$. Observe that the kernel of ϕ is contained in P. Indeed, if $g \in \ker\phi$, then in particular $gP = P$; hence $g \in P$. There are two choices: either $\ker\phi$ is equal to $\{\text{id}\}$ or $\ker\phi = P$. If $\ker\phi = \{\text{id}\}$, then G is isomorphic to a twelve element subgroup of S_4. Apply Lemma 2.83 to conclude that G is isomorphic to A_4.

Henceforth, $\ker\phi = P$. It follows, in particular, that $P \lhd G$; and therefore, P is the only Sylow 3-subgroup of G. It follows that G has exactly 2 elements of order 3. Let c be one of the elements of G of order 3. Every element of the conjugacy class of c, which is $\{gcg^{-1}|g \in G\}$, has order 3. Thus the conjugacy class of c has one or two elements. Of course, the size of the conjugacy class of c is $[G : \operatorname{stab} c]$. Thus $\operatorname{stab} c$ has either 6 or 12 elements. Recall that $\operatorname{stab} c$ is called the centralizer of c and this group is equal to $\{g \in G|gc = cg\}$. The centralizer of c is a group whose order is divisible 2. Cauchy's Theorem ensures that there is an element d of order 2 which commutes with c. The element $a = cd$ has order 6. (See Lemma 2.30, if necessary.) The subgroup $\langle a\rangle$ of G has index 2; consequently, $\langle a\rangle$ is a normal subgroup of G. Take $b \in G \setminus \langle a\rangle$. It follows that $b^2 \in \langle a\rangle$ and $bab^{-1} \in \langle a\rangle$. But we know much more. The element bab^{-1} must have order 6; so the only choices for bab^{-1} are a and a^5. Furthermore, bab^{-1} can not equal a because the group G is not Abelian. In a similar manner, we observe that b^2 can not equal a or a^5, because in either of these cases $\langle a\rangle$ would be a proper subgroup of $\langle b\rangle$:

$$6 = |\langle a\rangle| < |\langle b\rangle| \le 12.$$

Lagrange's Theorem would force $G = \langle b\rangle$, which is not possible because G is not Abelian. If $b^2 = a^2$, then

$$a^4 = a^5a^5 = (bab^{-1})(bab^{-1}) = ba^2b^{-1} = bb^2b^{-1} = b^2 = a^2;$$

and this contradicts the fact that a has order 6. Similarly, if $b^2 = a^4$, then

$$a^2 = (a^5)^4 = (bab^{-1})^4 = ba^4b^{-1} = bb^2b^{-1} = b^2 = a^4,$$

which is still impossible. Thus, $b^2 = \mathrm{id}$ and G is D_6; or $b^2 = a^3$ and G is the dicyclic group. □

2.I.c. *A list of groups of small order.* Every group of order n is isomorphic to exactly one of the groups in the second column.

order	the groups	explanation
1	$\{\mathrm{id}\}$	
2	$\frac{\mathbb{Z}}{2\mathbb{Z}}$	Use Lagrange's Theorem.
3	$\frac{\mathbb{Z}}{3\mathbb{Z}}$	Use Lagrange's Theorem.
4	$\frac{\mathbb{Z}}{4\mathbb{Z}}, \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$	See Corollary 2.74.
5	$\frac{\mathbb{Z}}{5\mathbb{Z}}$	Use Lagrange's Theorem.
6	$\frac{\mathbb{Z}}{6\mathbb{Z}}, S_3$	See Homework problem 15.
7	$\frac{\mathbb{Z}}{7\mathbb{Z}}$	Use Lagrange's Theorem.
8	$\frac{\mathbb{Z}}{8\mathbb{Z}}, \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}, \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}, D_4, Q_8$	See Homework problem 20 and Theorem 2.96.
9	$\frac{\mathbb{Z}}{9\mathbb{Z}}, \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$	See Corollary 2.74.
10	$\frac{\mathbb{Z}}{10\mathbb{Z}}, D_5$	See Homework problem 15.
11	$\frac{\mathbb{Z}}{11\mathbb{Z}}$	Use Lagrange's Theorem.
12	$\frac{\mathbb{Z}}{12\mathbb{Z}}, \frac{\mathbb{Z}}{6\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}, D_6, A_4$, the dicyclic group	See Theorems 2.88 and 2.96.
13	$\frac{\mathbb{Z}}{13\mathbb{Z}}$	Use Lagrange's Theorem.
14	$\frac{\mathbb{Z}}{14\mathbb{Z}}, D_7$	See Homework problem 15.
15	$\frac{\mathbb{Z}}{15\mathbb{Z}}$	See Observation 2.82
16	There are 14 groups of order 16.	See [12].
17	$\frac{\mathbb{Z}}{17\mathbb{Z}}$	Use Lagrange's Theorem.
18	There are 5 groups of order 18.	See Theorem 2.89
19	$\frac{\mathbb{Z}}{19\mathbb{Z}}$	Use Lagrange's Theorem.

Theorem 2.89. *If G is a group of order* 18*, then G is isomorphic to exactly one of the groups* $\frac{\mathbb{Z}}{18\mathbb{Z}}$, $\frac{\mathbb{Z}}{6\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$, D_9, $S_3 \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$, *or* $\left(\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}\right) \rtimes_\theta \frac{\mathbb{Z}}{2\mathbb{Z}}$, *where* $\theta : \frac{\mathbb{Z}}{2\mathbb{Z}} \to \operatorname{Aut}\left(\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}\right)$ *sends* $\bar{1}$ *from* $\frac{\mathbb{Z}}{2\mathbb{Z}}$ *to the automorphism of* $\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$ *which sends each element to its inverse.*

Proof. Let G be a group of order 18. If G is Abelian, then G is isomorphic to either $\frac{\mathbb{Z}}{18\mathbb{Z}}$ or $\frac{\mathbb{Z}}{6\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$. (Either make a direct proof or appeal to the structure theorem.) Henceforth, assume that G is not Abelian. The Sylow Theorem guarantees that G has a subgroup N of order 9. This subgroup has index 2 in G; so it is a normal subgroup of G. Every group of order p^2 is Abelian, so N is Abelian.

Let Z be the center of G.

Claim 2.89.1. *The order of Z is* 3 *or* 1.

Proof of Claim 2.89.1. If b is any element of G not in N, then $G = N \cup bN$. If b were in Z, then G would be Abelian. Thus, $b \notin Z$ and $Z \subseteq N$. The group Z can not be all of N; or else, once again, G would be Abelian. It follows that Z is a proper subgroup of N. Claim 2.89.1 is established.

Claim 2.89.2. *If* $|Z| = 3$*, then* $G \cong S_3 \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$.

Proof of Claim 2.89.2. In this case $\frac{G}{Z}$ is isomorphic to $\frac{\mathbb{Z}}{6\mathbb{Z}}$ or S_3. However, if $\frac{G}{Z}$ were isomorphic to $\frac{\mathbb{Z}}{6\mathbb{Z}}$, then G would be Abelian and this case has been ruled out of consideration. So, $\frac{G}{Z}$ must be isomorphic to S_3. Thus, there exist a and b in G with $a^3 \in Z$, $b^2 \in Z$, $abab \in Z$, and G is generated by a, b, and Z.

Observe that there exist A and B in G so that $A^3 = \mathrm{id}$, $B^2 = \mathrm{id}$, $ABAB = \mathrm{id}$, and G is generated by A, B, and Z. Indeed, if $b^2 = c \in Z$, then take $B = bc$. Observe that $B^2 = bcbc = bbcc = c^3 = \mathrm{id}$. If $(aB)^2 = c' \in Z$, then take $A = c'a$. Observe that $ABAB = c'aBc'aB = c'c'(aB)^2 = c'c'c' = \mathrm{id}$. Observe that A^3 must equal id. Indeed, let us suppose $A^3 = z \in Z$. We already showed that $BAB^{-1} = A^{-1}$; hence,

$$z = BzB^{-1} = BA^3B^{-1} = A^{-3} = z^{-1}.$$

The only element of Z which is its own inverse is id.

Observe that the inclusion maps induce an isomorphism

$$Z \oplus <A, B> \to G,$$

and Claim 2.89.2 is established.

Claim 2.89.3. *If* $|Z| = 1$ *and x is any element of G with* $x \notin N$*, then* $x^2 = \mathrm{id}$.

Proof of Claim 2.89.3. The group $\frac{G}{N}$ is cyclic of order 2; so, $x^2 \in N$. Thus, x^2 commutes with x and also with every element of N. It follows that $x^2 \in Z = \{\mathrm{id}\}$. The proof of Claim 2.89.3 is complete.

Claim 2.89.4. *If* $|Z| = 1$ *and N is cyclic, then* $G \cong D_9$.

Proof of Claim 2.89.4. Fix a generator a for N and any element $b \in G \setminus N$. Observe that G is a group of 18 elements generated by a, b with $a^9 = \mathrm{id}$, $b^2 = \mathrm{id}$ (take $x = b$ in Claim 2.89.3) and $(ab)^2 = \mathrm{id}$ (take $x = ab$ in Claim 2.89.3). Thus, G is isomorphic D_9 by Theorem 2.61.1 and the proof of Claim 2.89.4 is complete.

Claim 2.89.5. *If* $|Z| = 1$ *and* $N \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$, *then* G *is isomorphic to* $\left(\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}\right) \rtimes_\theta \frac{\mathbb{Z}}{2\mathbb{Z}}$ *as described above.*

Proof of Claim 2.89.5. Fix generators a, b for N. Pick $c \in G \setminus N$. Observe that G is generated by a, b, c. Observe also that $a^3 = b^3 = \mathrm{id}$ and $ab = ba$. Apply Claim 2.89.3 three times to see that $c^2 = (ca)^2 = (cb)^2 = \mathrm{id}$. Consider the free group $F = \langle X, Y, Z\rangle$. Let N be the smallest normal subgroup of F which contains $X^3, Y^3, Z^2, (ZX)^2, (ZY)^2, XYX^2Y^2$. It is clear that $\frac{F}{N}$ has at most 18 elements. If $\frac{F}{N}$ has at least 18 elements then there is a surjective homomorphism $\frac{F}{N} \to G$ which sends the class of X to a, the class of Y to b and the class of Z to c and $G \cong \frac{F}{N}$.

We finish the proof by exhibiting a surjection from $\frac{F}{N}$ onto a group with 18 elements that is known to exist. This surjection shows that $|\frac{F}{N}| \geq 18$; and therefore, the calculation of the previous paragraph may be made. Of course, $\left(\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}\right) \rtimes_\theta \frac{\mathbb{Z}}{2\mathbb{Z}}$ is an honest group with 18 elements. Take $a = ((1,0),0)$, $b = ((0,1),0)$, $c = ((0,0),1)$. There is no difficulty observing that $a^3 = b^3 = c^2 = ((0,0),0)$;

$$\begin{aligned} ca &= ((0,0),1)((1,0),0) = ((-1,0),1),\\ cb &= ((0,0),1)((0,1),0) = ((0,-1),1),\\ (ca)^2 &= ((-1,0),1)((-1,0),1) = ((0,0),0) = (cb)^2 \end{aligned}$$

and $ab = ba$. The proof of Claim 2.89.5 is complete.

We have shown that if G is a non-Abelian group of order 18, then G is isomorphic to D_9, $S_3 \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$, or $\left(\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}\right) \rtimes_\theta \frac{\mathbb{Z}}{2\mathbb{Z}}$, where $\theta : \frac{\mathbb{Z}}{2\mathbb{Z}} \to \mathrm{Aut}\left(\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}\right)$ sends $\bar{1}$ from $\frac{\mathbb{Z}}{2\mathbb{Z}}$ to the automorphism of $\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$ which sends each element to its inverse. It is clear that none of these three groups isomorphic to any other group from the list. The center of $S_3 \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$ has 3 elements; the centers of $S_3 \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$ and $\left(\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}\right) \rtimes_\theta \frac{\mathbb{Z}}{2\mathbb{Z}}$ each have 1 element. The Sylow 3-subgroup of D_9 is cyclic; but the Sylow 3-subgroup of $\left(\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}\right) \rtimes_\theta \frac{\mathbb{Z}}{2\mathbb{Z}}$ is not cyclic. □

2.J. **Finitely generated Abelian groups.** The ultimate theorem (Theorem 2.96) is obtained from Theorem 2.90 by way of multiple uses of the Chinese Remainder Theorem (Example 2.60). Indeed, Theorem 2.90 is the main result; Theorem 2.96 is merely Theorem 2.90 with decorations painted on it.

Recall that the Abelian group G is finitely generated if there exist an integer n and a surjective group homomorphism $\phi : \mathbb{Z}^n \to G$.

Theorem 2.90. *Every finitely generated Abelian group is isomorphic to the direct sum of cyclic groups.*

Theorem 2.90 is a consequence of the following five results.

Lemma 2.91. *Every subgroup of $\mathbb{Z}^n$ is generated by n or fewer generators.*

Corollary 2.92. *If G is a finitely generated Abelian group then there exist non-negative integers m and n and an $n \times m$ matrix of integers M such that*

$$\frac{\mathbb{Z}^n}{\textit{the subgroup of } \mathbb{Z}^n \textit{ generated by the columns of } M} \cong G. \tag{2.92.1}$$

Remark 2.92.2. The Abelian group on the left side of (2.92.1) is usually called the cokernel of M and denoted coker M.

Lemma 2.93. *If $M_{n\times m}$, $N_{n\times n}$, and $P_{m\times m}$ are matrices of integers with N and P invertible over $\mathbb{Z}$, then* $\operatorname{coker} M \cong \operatorname{coker}(NMP)$.

Lemma 2.94. *If $M_{n\times m}$ is a matrix of integers, then there exist matrices $N_{n\times n}$ and $P_{m\times m}$, which are invertible over $\mathbb{Z}$, such that*

$$NMP = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right],$$

where D equal to the diagonal matrix

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_r \end{bmatrix},$$

with $d_i \neq 0$.

Lemma 2.95. *If*

$$M' = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right],$$

with D equal to the diagonal matrix

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_r \end{bmatrix},$$

is an $n \times m$ matrix of integers, then

$$\operatorname{coker} M' = \frac{\mathbb{Z}}{d_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{d_r\mathbb{Z}} \oplus \mathbb{Z}^{n-r}.$$

Lemma. 2.91. *Every subgroup of $\mathbb{Z}^n$ is generated by n or fewer generators.*

Remark. This is a special case of the result "every finitely generated module over a Noetherian ring is Noetherian".

Proof. The proof is by induction on n. We already proved that every subgroup of $\mathbb{Z}$ is cyclic; see Proposition 2.24.

Let G be a subgroup of $\mathbb{Z}^n$. Let

$$G_1 = \left\{ r \in \mathbb{Z} \,\middle|\, \exists b \in \mathbb{Z}^{n-1} \text{ with } \begin{bmatrix} r \\ b \end{bmatrix} \in G \right\}$$

and

$$G_2 = \left\{ b \in \mathbb{Z}^{n-1} \,\middle|\, \begin{bmatrix} 0 \\ b \end{bmatrix} \in G \right\}.$$

Observe that G_1 is a subgroup of $\mathbb{Z}$ and G_2 is a subgroup of $\mathbb{Z}^{n-1}$. Thus, G_1 is a cyclic group, and, by induction G_2 can be generated by $n-1$ elements. Let $b_2, \ldots, b_n$ be a generating set for G_2 and r_1 be a generator of G_1. There exists $b_1 \in \mathbb{Z}^{n-1}$ with $\begin{bmatrix} r_1 \\ b_1 \end{bmatrix} \in G$. Observe that

$$\begin{bmatrix} r_1 \\ b_1 \end{bmatrix}, \begin{bmatrix} 0 \\ b_2 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ b_n \end{bmatrix}$$

generates G. □

Corollary. 2.92. *If G is a finitely generated Abelian group, then there exist non-negative integers m and n and an $n \times m$ matrix of integers M such that*

$$\frac{\mathbb{Z}^n}{\textit{the subgroup of } \mathbb{Z}^n \textit{ generated by the columns of } M} \cong G.$$

Proof. The hypothesis that G is a finitely generated Abelian group guarantees that there is a surjective homomorphism

$$\mathbb{Z}^n \xrightarrow{\pi} G.$$

The First Isomorphism Theorem yields that

$$G \cong \frac{\mathbb{Z}^n}{\ker \pi}.$$

Apply Lemma 2.91 to see that $\ker \pi$ is a finitely generated subgroup of $\mathbb{Z}^n$. Take a generating set for $\ker \pi$ and arrange this generating set to be the columns of a matrix. □

Lemma. 2.93. *If $M_{n\times m}$, $N_{n\times n}$, and $P_{m\times m}$ are matrices of integers with N and P invertible over $\mathbb{Z}$, then* $\operatorname{coker} M \cong \operatorname{coker}(NMP)$.

Proof. Consider the following commutative diagram of homomorphisms of Abelian groups

$$\begin{tikzcd}
\mathbb{Z}^m \arrow[r, "M"] \arrow[d, "P^{-1}"', "\cong"] & \mathbb{Z}^n \arrow[r, "q"] \arrow[d, "N", "\cong"'] & \operatorname{coker} M \arrow[r] & 0 \\
\mathbb{Z}^m \arrow[r, "NMP"] & \mathbb{Z}^n \arrow[r, "q'"] & \operatorname{coker}(NMP) \arrow[r] & 0,
\end{tikzcd}$$

where q and q' are the natural quotient maps. The composition $q' \circ N$ is a surjective group homomorphism $\mathbb{Z}^n \to \operatorname{coker}(NMP)$. Apply the First Isomorphism Theorem to see that $q' \circ N$ induces an isomorphism

$$\overline{(q' \circ N)} : \frac{\mathbb{Z}^n}{\ker(q' \circ N)} \longrightarrow \operatorname{coker}(NMP).$$

We show that

$$\ker(q'\circ N) = \operatorname{im} M.$$

Of course, that completes the proof since

$$\frac{\mathbb{Z}^n}{\operatorname{im} M} = \operatorname{coker} M.$$

The inclusion $\operatorname{im} M \subseteq \ker(q'\circ N)$ is obvious because

$$q'\circ N\circ M = q'\circ(NMP)\circ P^{-1}$$

and the kernel of q' is equal to the image of NMP.

Now we prove $\ker(q'\circ N) \subseteq \operatorname{im} M$. Let $x \in \ker(q'\circ N)$. It follows that

$$Nx \in \ker q' = \operatorname{im}(NMP).$$

Thus, there exists an element $y \in \mathbb{Z}^m$ with $Nx = NMPy$. The matrix N is invertible; hence $x = MPy \in \operatorname{im} M$. □

Lemma. 2.94. *If $M_{n\times m}$ is a matrix of integers, then there exist matrices $N_{n\times n}$ and $P_{m\times m}$, which are invertible over $\mathbb{Z}$, such that*

$$NMP = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right],$$

where D equal to the diagonal matrix

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_r \end{bmatrix},$$

with $d_i \neq 0$.

Proof. We apply a sequence of elementary row and column operations, which are invertible over $\mathbb{Z}$, to M in order to produce a matrix whose only non-zero entries live on the main diagonal. Notice that there are six elementary row and column operations which are invertible over $\mathbb{Z}$, namely:

(1) we may exchange two rows,
(2) we may exchange two columns,
(3) we may add an integer multiple of one row to a different row,
(4) we may add an integer multiple of one column to a different column,
(5) we may multiply any row by -1, and
(6) we may multiply any column by -1.

The proof is by induction. We will apply elementary operations, as described above, until we obtain a matrix with every entry in row one and column one, except possibly the entry in position $(1, 1)$, equal to zero. Then we are finished by induction.

Step A. If every entry in row 1 and column 1 is zero, then we are finished.

Step B. If some entry in row 1 or column 1 is non-zero then we apply elementary operations in order to make the $(1, 1)$ entry be positive.

Step C. If the $(1,1)$ entry divides every entry in row 1 and column 1, then we apply elementary row and column operations and turn all of the entries in row 1 and column 1 other than the (1,1) entry into zero. We are finished.

Step D. The only remaining possibility is that the $(1,1)$ entry $x_{1,1}$ does not divide $x_{1,j}$ for some j (or $x_{i,1}$ for some i). In this case, we use the division algorithm for integers. The integer $x_{1,1}$ is positive; consequently, there exist integers q and r with

$$x_{1,j} = qx_{1,1} + r \quad \text{or} \quad x_{i,1} = qx_{1,1} + r$$

and $1 \le r \le x_{1,1} - 1$. We apply two elementary operations in order to put a smaller positive entry in position $(1,1)$. That is, we replace column j with column j minus q times column 1 and then we exchange column 1 and column j. (Or we replace row i with row i minus q times row 1 and then we exchange row 1 and row j.) Return to Step C.

The process stops after a finite number of iterations. □

Lemma. 2.95. *If*

$$M' = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right],$$

with D equal to the diagonal matrix

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_r \end{bmatrix},$$

is an $n \times m$ matrix of integers, then

$$\operatorname{coker} M' = \frac{\mathbb{Z}}{d_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{d_r\mathbb{Z}} \oplus \mathbb{Z}^{n-r}.$$

Proof. Consider the group homomorphism

$$\phi : \mathbb{Z}^n \to \frac{\mathbb{Z}}{d_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{d_r\mathbb{Z}} \oplus \mathbb{Z}^{n-r},$$

which is given by

$$\phi\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\right) = \left(\bar{a}_1, \dots, \bar{a}_r, \begin{bmatrix} a_{r+1} \\ \vdots \\ a_n \end{bmatrix}\right).$$

Apply the First Isomorphism Theorem:

$$\frac{\mathbb{Z}^n}{\ker \phi} \cong \frac{\mathbb{Z}}{d_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{d_r\mathbb{Z}} \oplus \mathbb{Z}^{n-r}.$$

Observe that $\ker \phi$ is generated by

$$\begin{bmatrix} d_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ d_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ d_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

(In the vector on the right, d_r appears in row r.) □

Theorem 2.96. *Let G be a finite Abelian group. Then the following statements hold.*

(a) *There exist positive prime integers p_i and positive integers e_i such that*

$$G \cong \bigoplus_i \frac{\mathbb{Z}}{p_i^{e_i}\mathbb{Z}}. \tag{2.96.1}$$

Furthermore, the decomposition of (2.96.1) *is unique in the sense that if q_j are positive prime integers and f_j are positive integers with*

$$G \cong \bigoplus_j \frac{\mathbb{Z}}{q_j^{f_j}\mathbb{Z}},$$

then each decomposition has the same number of factors and, after renumbering, $p_i = q_i$ and $e_i = f_i$, for all i.

(b) *There exist positive integers $\lambda_1, \ldots \lambda_r$ such that*

$$G \cong \frac{\mathbb{Z}}{\lambda_1\mathbb{Z}} \oplus \ldots \oplus \frac{\mathbb{Z}}{\lambda_r\mathbb{Z}} \quad \textit{and} \quad \lambda_1|\lambda_2| \cdots |\lambda_r. \tag{2.96.2}$$

Furthermore, this decomposition is completely unique; if $\mu_1, \ldots, \mu_s$ are positive integers with

$$G \cong \frac{\mathbb{Z}}{\mu_1\mathbb{Z}} \oplus \ldots \oplus \frac{\mathbb{Z}}{\mu_s\mathbb{Z}} \quad \textit{and} \quad \mu_1|\mu_2| \cdots |\mu_s,$$

then $r = s$ and $\lambda_i = \mu_i$ for all i.

Proof.

The existence of decomposition (2.96.1). The decomposition of (2.96.1) is obtained by applying the Chinese Remainder Theorem (Example 2.60) to the decomposition of Theorem 2.90:

$$G \cong \frac{\mathbb{Z}}{d_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{d_r\mathbb{Z}}.$$

If $d = p_1^{e_1} \cdots p_\ell^{e_\ell}$, where the p_i are distinct positive prime integers and the e_i are prime integers, then

$$\frac{\mathbb{Z}}{d\mathbb{Z}} = \frac{\mathbb{Z}}{p_1^{e_1}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_\ell^{e_\ell}\mathbb{Z}}.$$

The existence of decomposition (2.96.2). Begin with the decomposition of (2.96.1). Arrange the summands of G by using "right justification". That is, identify the positive prime integers $p_1, \ldots, p_s$ which contribute a summand to G. For each p_i identify the corresponding exponents

$$0 \leq e_{i,1} \leq e_{i,2} \leq \cdots \leq e_{i,r} \tag{2.96.3}$$

Notice that in (2.96.3) all strings of exponents have the same length. We accomplished this but putting zeros in front of each short exponent string. Notice that $\frac{\mathbb{Z}}{p_i^0\mathbb{Z}}$ is the group $\{0\}$. It does no harm to include zero as a direct summand.

$$G = \begin{cases} \frac{\mathbb{Z}}{p_1^{e_{11}}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p_1^{e_{12}}\mathbb{Z}} \oplus \dots \oplus \frac{\mathbb{Z}}{p_1^{e_{1r}}\mathbb{Z}} \\ \oplus \frac{\mathbb{Z}}{p_2^{e_{21}}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p_2^{e_{22}}\mathbb{Z}} \oplus \dots \oplus \frac{\mathbb{Z}}{p_2^{e_{2r}}\mathbb{Z}} \\ \vdots \\ \oplus \frac{\mathbb{Z}}{p_s^{e_{s1}}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p_s^{e_{s2}}\mathbb{Z}} \oplus \dots \oplus \frac{\mathbb{Z}}{p_s^{e_{sr}}\mathbb{Z}} \end{cases}$$

Let $\lambda_j = \prod_i p_i^{e_{ij}}$. (So λ_j is the product of the generator of the denominators that appear in column j.) Apply the Chinese Remainder Theorem to see that

$$G \cong \frac{\mathbb{Z}}{\lambda_1\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\lambda_2\mathbb{Z}} \oplus \dots \oplus \frac{\mathbb{Z}}{\lambda_r\mathbb{Z}}.$$

It is clear from the construction that $\lambda_1|\lambda_2|\dots|\lambda_r$.

Now we show that the p-primary decomposition is unique. This argument consists of a few steps.

Observation 2.97. *Let G_1 and G_2 be finite Abelian groups and n_1 and n_2 be non-negative integers. If*

$$G_1 \oplus \mathbb{Z}^{n_1} \quad \textit{and} \quad G_2 \oplus \mathbb{Z}^{n_2}$$

are isomorphic Abelian groups, then $G_1 \cong G_2$ and $n_1 = n_2$.

Proof. If G is an Abelian then the torsion subgroup of G is

$$\tau(G) = \{g \in G \mid \text{ there exists a positive integer } N \text{ with } Ng = 0\}.$$

Notice that if

$$G_1 \oplus \mathbb{Z}^{n_1} \cong G_2 \oplus \mathbb{Z}^{n_2},$$

then $\tau(\text{LHS}) \cong \tau(\text{RHS})$ (hence $G_1 \cong G_2$) and any isomorphism $\phi : \text{LHS} \to \text{RHS}$ satisfies

$$\phi(\tau(\text{LHS})) = \tau(\text{RHS}).$$

Apply the First Isomorphism Theorem to

$$\phi : \text{LHS} \to \frac{\text{RHS}}{\tau(\text{RHS})}$$

to conclude

$$\frac{\text{LHS}}{\tau(\text{LHS})} \cong \frac{\text{RHS}}{\tau(\text{RHS})};$$

hence,

$$\mathbb{Z}^{n_1} \cong \mathbb{Z}^{n_2};$$
$$\frac{\mathbb{Z}^{n_1}}{2\mathbb{Z}^{n_1}} \cong \frac{\mathbb{Z}^{n_2}}{2\mathbb{Z}^{n_2}};$$

and

$$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{n_1} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{n_2}.$$

Count the number of elements to conclude $n_1 = n_2$. □

The uniqueness of decomposition (2.96.1). Suppose the p_i and q_j are distinct positive prime integers, the e_i and f_j are positive integers, and

$$\bigoplus_i \frac{\mathbb{Z}}{p_i^{e_i}\mathbb{Z}} \cong \bigoplus_j \frac{\mathbb{Z}}{q_j^{f_j}\mathbb{Z}}. \tag{2.97.1}$$

We want to prove that both decompositions have the same number of factors and that, after renumbering, $p_i = q_i$ and $e_i = f_i$. Let p be a prime integer. The p-primary subgroup of G is the

$$\{g \in G \mid p^N g = 0 \text{ for some positive integer } N\}.$$

Observe that if p and q are positive prime integers then the p-primary subgroup of

$$\frac{\mathbb{Z}}{q^f\mathbb{Z}} = \begin{cases} 0 & \text{if } p \neq q, \\ \frac{\mathbb{Z}}{q^f\mathbb{Z}} & \text{if } p = q. \end{cases}$$

Indeed, it is clear that $p^N \frac{\mathbb{Z}}{p^f\mathbb{Z}} = 0$ for all $f \le N$. It is also clear that p^N acts like a unit on $\frac{\mathbb{Z}}{q^f\mathbb{Z}}$ if $p \neq q$ because there exist integers[31] a and b with $ap^N + bq^f = 1$; so ap^N acts like $1 - bq^f$ on $\frac{\mathbb{Z}}{q^f\mathbb{Z}}$ and a acts like the inverse of p^N on $\frac{\mathbb{Z}}{q^f\mathbb{Z}}$.

Consider the p-primary component of (2.97.1) for each positive prime integer p. It suffices to prove that if

$$\frac{\mathbb{Z}}{p^{e_1}\mathbb{Z}} \oplus \dots \oplus \frac{\mathbb{Z}}{p^{e_r}\mathbb{Z}} \cong \frac{\mathbb{Z}}{p^{f_1}\mathbb{Z}} \oplus \dots \oplus \frac{\mathbb{Z}}{p^{f_s}\mathbb{Z}} \tag{2.97.2}$$

with

$$1 \le e_1 \le \cdots \le e_r \quad \text{and} \quad 1 \le f_1 \le \cdots \le f_s,$$

then $r = s$ and $e_i = f_i$ for all i. We use two tricks to finish the argument.

Trick one. If G is an Abelian group and p is an integer, then let

$$\{0\} :_G p = \{g \in G | pg \in \{0\}\}.$$

Observe that

$$|\{0\} :_{\text{LHS}} p| = p^r \quad \text{and} \quad |\{0\} :_{\text{RHS}} p| = p^s$$

because the subgroup of $\frac{\mathbb{Z}}{p^e\mathbb{Z}}$ of elements of order p or less is generated by $\overline{p^{e-1}}$. There are p elements in this subgroup:

$$\overline{p^{e-1}},\ \overline{2p^{e-1}},\ \overline{3p^{e-1}},\ \dots,\ \overline{(p-1)p^{e-1}}.$$

So, $\boxed{r = s}$ and "we continue in this manner to finish the argument".

Trick two. One way to "continue in this manner" is to "throw away" all of the summands of the form $\frac{\mathbb{Z}}{p\mathbb{Z}}$. One very clean way to do this is to look at the subgroup pG of G. Of course,

$$pG = \{\underbrace{g + \cdots + g}_{p} | g \in G\}.$$

[31] Use Lemma 2.60.1, if necessary.

Notice that if $G = \frac{\mathbb{Z}}{p^e\mathbb{Z}}$, then

$$\begin{cases} pG = \{0\} & \text{if } e = 1 \\ pG \cong \frac{\mathbb{Z}}{p^{e-1}\mathbb{Z}} & \text{if } 2 \leq e. \end{cases}$$

The assertion when $e = 1$ is obvious. Use the First Isomorphism Theorem to prove the assertion when $2 \leq e$. Indeed, if $(G, +)$ is any Abelian group, then there is a surjective homomorphism

$$\phi : G \to pG$$

given by $\phi(g) = \underbrace{g + \cdots + g}_{p}$. Observe that $\ker \phi = \{0\} :_G p$. Apply the First Isomorphism Theorem to

$$\phi : \frac{\mathbb{Z}}{p^e\mathbb{Z}} \to p\frac{\mathbb{Z}}{p^e\mathbb{Z}},$$

given by $\phi(g) = pg$, to obtain

$$\frac{\frac{\mathbb{Z}}{p^e\mathbb{Z}}}{\ker \phi} \cong p\frac{\mathbb{Z}}{p^e\mathbb{Z}}.$$

Recall that we already observed that

$$\ker \phi = \{0\} :_G p,$$

and, if $G = \frac{\mathbb{Z}}{p^e\mathbb{Z}}$, then $\{0\} :_G p = p^{e-1}\frac{\mathbb{Z}}{p^e\mathbb{Z}}$. Conclude

$$\frac{\frac{\mathbb{Z}}{p^e\mathbb{Z}}}{p^{e-1}\frac{\mathbb{Z}}{p^e\mathbb{Z}}} \cong p\frac{\mathbb{Z}}{p^e\mathbb{Z}}.$$

Use the second isomorphism theorem to see that the group on the left is

$$\frac{\frac{\mathbb{Z}}{p^e\mathbb{Z}}}{\frac{p^{e-1}\mathbb{Z}}{p^e\mathbb{Z}}} \cong \frac{\mathbb{Z}}{p^{e-1}\mathbb{Z}}.$$

Multiply both sides of (2.97.2) by p and calculate $|\{0\} :_{\square} p|$ to see that

$$|0 :_{p\text{LHS}} p| = |0 :_{p\text{RHS}} p|;$$

hence

$$p^{|\{i|2\leq e_i\}|} = p^{|\{i|2\leq f_i\}|}.$$

Thus,

$$|\{i|2 \leq e_i\}| = |\{i|2 \leq f_i\}|$$

and

$$\boxed{|\{i|1 = e_i\}| = |\{i|1 = f_i\}|}$$

and the proof is completed by induction (or by iteration).

The uniqueness of the decomposition (2.96.1) implies the uniqueness of the decomposition (2.96.2).

□

3. Rings

3.A. The basics.

Definition 3.1. The set R with two operations $+$ and $\cdot$ is a ring if

(a) $(R, +, 0)$ is an Abelian group,
(b) $r \cdot r' \in R$,
(c) there exists $1 \in R$ with $1 \cdot r = r = r \cdot 1$,
(d) $r \cdot (r' \cdot r'') = (r \cdot r') \cdot r''$,
(e) $r(s + s') = rs + rs'$, and
(f) $(s + s')r = sr + s'r$ for all r, r', r'', s, s' in R.

Examples 3.2.
- The set of integers $\mathbb{Z}$ under addition and multiplication is a ring.
- Every field is a ring.
- If R is a ring, then $R[x]$ (the set of polynomials in one variable with coefficients in R) is a ring.
- If R is a ring, then $R[[x]]$ (the set of formal power series in one variable with coefficients in R) is a ring.
- If $R \subseteq S$ are rings and $s_1, \dots$ are elements of S, then $R[s_1, \dots]$ is the smallest subring of S that contains R and $s_1, \dots$, (For example $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[i]$, and $\mathbb{Q}[\pi]$ are subrings of $\mathbb{C}$.)
- If R is a ring, then $\mathrm{Mat}_{n \times n}(R)$ (the set of $n \times n$ matrices with entries from R) is a ring.
- The set of continuous functions from $[0, 1]$ to $\mathbb{R}$ is a ring.

Words 3.3.
- The ring R is commutative if $rr' = r'r$ for all $r, r' \in R$.
- The ring R is a domain if R is commutative, $1 \neq 0$, and
$$ab = 0 \Rightarrow a = 0 \text{ or } b = 0.$$
- The ring R is a field if R is a commutative ring, $1 \neq 0$, and every non-zero element of R has a multiplicative inverse. (That is, if $r \in R \setminus \{0\}$, then there exists $r' \in R$ with $rr' = 1 = r'r$.)
- A division ring or skew field is a non-commutative ring R with $1 \neq 0$ and every non-zero element has a multiplicative inverse.

November 15, 2023

- characteristic
- module
- Examples of Division Rings
- Group rings
- check that R/I is a legitimate ring when I is a (two-sided) ideal of the ring R
- examples of ideals

Characteristic

Definition 3.4. If R is a ring then there is a ring homomorphism $\phi : \mathbb{Z} \to R$ with $\phi(1) = 1$. The kernel of ϕ is generated by a non-negative integer c. This c is called the characteristic of R.

Examples 3.5. Every ring that contains $\mathbb{Z}$ has characteristic zero. Our undergraduate students like characteristic p because if a and b are elements of a ring of characteristic p, then

$$(a+b)^p = a^p + \binom{p}{1} a^{p-1} b + \dots \binom{p}{p-1} ab^{p-1} + b^p = a^p + b^p.$$

So, in particular, the function $\phi : R \to R$, which is given by $\phi(r) = r^p$ is a ring homomorphism.[32]

Modules

Definition 3.6. Let R be a ring and M be an Abelian group. If there is a function $R \times M \to M$, which sends the ordered pair (r, m), with $r \in R$ and $m \in M$, to an element rm in M which satisfies

- $r(m_1 + m_2) = rm_1 + rm_2$,
- $(r_1 + r_2)m = r_1 m + r_3 m$,
- $(r_1 r_2)m = r_1(r_2 m)$,
- $1(m) = m$,

then M is a left R-module.

Examples 3.7. Let R be a ring.

- R is a left R-module.
- If M_i is a left R-module for all $i \in I$, then $\bigoplus_{i\in I} M_i$ is a left R-modules. In particular, $\bigoplus_{i\in I} R$ is a left R-module (called a free R-module).
- Every left ideal of R is a left R-module.
- If $N \subseteq M$ are left R-modules then the (well understood) Abelian group M/N is a left R-module, with scalar multiplication r times $m + N$ is equal to $rm + N$. (I guess we better check that this makes sense.)

[32] A ring homomorphism is a function ϕ from the ring R to the ring S for which $\phi(r + r') = \phi(r) + \phi(r')$, $\phi(rr') = \phi(r)\phi(r')$, and $\phi(1) = 1$ for all r and r' in R.

3.B. **Ideals, Quotient rings, and the First Isomorphism Theorem.**

Definition 3.8. Let R be a ring.

- The subset I of R is a left ideal is a subgroup of the Abelian group $(R,+,0)$ which is closed under left multiplication by elements of R.
- The subset I of R is a right ideal is a subgroup of the Abelian group $(R,+,0)$ which is closed under right multiplication by elements of R.
- The subset I of R is a two-sided ideal or ideal if I is both a left ideal and a right ideal of R.

Remark. If R is a commutative ring, then the concepts "left ideal", "right ideal", "two-sided ideal", and "ideal" are identical. Any subset of R which is one of these concepts is all of the concepts.

Examples of Division Rings.[33]

The Quaternions Let K be a subfield of $\mathbb{R}$ and let

$$\mathbb{H} = K \oplus Ki \oplus Kj \oplus Kk.$$

Define multiplication on $\mathbb{H}$ by

$$i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$

Now we see why every non-zero element of $\mathbb{H}$ has an inverse. Observe that

$$(a + bi + cj + dk)(a - bi - cj - dk)$$
$$= \left\{\begin{array}{l} \quad (a^2 + b^2 + c^2 + d^2) \\ +(-ab + ab - cd + dc)i \\ +(-ac + bd + ca - db)j \\ +(-ad + bd - cb + da)k. \end{array}\right.$$

If $x = a + bi + cj + dk$ is not equal to zero then $a^2 + b^2 + c^2 + d^2$ is not zero (because $K \subseteq \mathbb{R}$) and x^{-1} is equal to

$$\frac{1}{a^2 + b^2 + c^2 + d^2}(a - bi - cj - dk).$$

Here are other similar Division rings. Let p be a prime integer which is congruent to 3 mod 4. Consider $H = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}k$. Define $i^2 = -1,\ j^2 = p,\ ij = -ji = k$. The inverse of $a + bi + cj + dk$ is

$$\frac{1}{a^2 + b^2 - pc^2 - pd^2}(a - bi - cj - dk).$$

Some Number Theory tricks guarantee that $a^2 + b^2 - pc^2 - pd^2 \neq 0$.

Endomorphism rings If M is a simple left module over the ring R, then $\mathrm{End}_R(M)$ is a division ring.

A simple module is a module M with no submodules other than 0 and M.

An R-module Endomorphism of M is an R-module homomorphism from M to M.

If M and N are R-modules, then an R-module homomorphism from M to N is a homomorphism of Abelian groups $\phi : M \to N$ which "respects" scalar multiplication in the sense that $\phi(rm) = r\phi(m)$ for all $r \in R$ and $m \in M$.

Of course, the image of an R-module homomorphism is an R-module. If the target is a simple module, then the homomorphism is either the zero map or is surjective.

Similarly, the kernel of an R-module homomorphism is an R-module. If the domain is a simple module, then the homomorphism is either the zero map or is injective.

It is now clear that every non-zero R-module homomorphism from a simple R-module M to itself is an isomorphism and therefore, $\mathrm{End}_R(M)$ is a division ring.

[33] I found this information at
https://ysharifi.wordpress.com/2022/03/25/examples-of-division-rings/

Example 3.9. If $\mathfrak{m}$ is a maximal left ideal in the ring R, then $R/\mathfrak{m}$ is a simple left R-module and $\operatorname{End}_R(R/\mathfrak{m})$ is a division ring.

In particular, if R is the ring of $n \times n$ matrices with entries from the field $\boldsymbol{k}$ and $\mathfrak{m}$ is the subset of R with every entry in column n equal to zero, then $\mathfrak{m}$ is a maximal left ideal of R. So $\operatorname{End}_R(R/\mathfrak{m})$ is a Division ring.

Total ring of fractions

Example 3.10. Let R be a Noetherian ring with no zero divisors. Let Q be the set

$$\{\tfrac{r}{s} \mid r \in R \text{ and } s \in R \setminus \{0\}\}.$$

Define $+, -, \times, \div$ in the obvious manner. (We will do this procedure slowly when R is commutative.) Then Q is a Division ring. (I skipped over something here. If you really care, check the details carefully.)

November 27, 2023

The Final exam is Friday Dec. 15, 4:00-6:30 in our usual class room.

The Final is comprehensive.

Questions 2, 3, 4 from Exam 2 were old Qual questions.

The question about 72 is hard; but now you have a new tool.

The question about $\frac{HN}{N}$ is easy.

The question about p^6 and p^7 is very standard. It can also be asked about Jordan canonical forms. The point is that to count the number of Abelian groups of order p^n with certain properties or to describe the set of JCF of $n \times n$ matrices with certain properties is equivalent to counting partitions of n with certain properties.

With respect to problem 1, Part (a) is a silly question. We have been thinking about the direct sum of Abelian groups. In an "Abelian category" direct sum is equivalent to has a "splitting map". This explains (c). The "category of groups" is not an "Abelian category". This explains (b).

When we last had class, we were listing examples of Division rings. We had $\mathbb{H}$ (and some twists), $\operatorname{End}_R(M)$ where M is a simple left R-module, and every Noetherian ring without zero divisors is naturally embedded in a smallest Division ring.

I want to give one more example.

Formal Laurent series.

If R is a ring, then the set of formal power series

$$R[[x]] = \{\sum_{i=0}^{\infty} r_i x^i \mid r_i \in R\}.$$

over R is another ring. If r_0 is a unit, then $\sum_{i=0}^{\infty} r_i x^i$ is also a unit. Indeed, the inverse of $1 - xp(x)$ is

$$\sum_{i=0}^{\infty} (xp(x))^i.$$

If R is a division ring, then every element of $R[[x]]$ has the form x^iunit for some i. We do not have to do much to turn $R[[x]]$ into a division ring; we only have to invert x. At any rate, the ring of Formal Laurent series

$$D((x)) = \left\{ \sum_{n \le i} d_i x^i \mid n \in \mathbb{Z} \text{ and } d_i \in D \right\}$$

is a division ring whenever D is a division ring.

Group rings

Definition 3.11. If R is a ring and G is a group, then the group ring $R[G]$ is a free R-module

$$\bigoplus_{g \in G} Rg.$$

The multiplication involving the g's is the multiplication from G.

Remark. The Quaternion ring $\mathbb{H}$ is inspired by the Quaternion group Q_8 but is NOT a group ring. (In fact, I suspect that $\mathbb{H} \cong \frac{K[Q_8]}{(a^2+1)}$, where Q_8 is the eight element group with elements $a^i b^j$, $0 \le i \le 3$,

$0 \le j \le 1$, $a^4 = \text{id}$, $a^2 = b^2$, and $ba = a^3b$. At any rate, a^2+1 is in the center of $K[Q_8]$; therefore, a^2+1 generates a two-sided ideal.)

Indeed, if G is a finite group then group rings $K[G]$ tends to have zero divisors. Indeed, if g is an element of G of order n, then

$$(1-g)(1+g+\cdots+g^{n-1}) = 1-g^n = 0.$$

Many rings and modules studied in Commutative Algebra or Algebraic Geometry are symmetric in the variables or are invariant under change of basis. If there are n variables involved then these rings and modules become $K[S_n]$-modules or $K[\text{GL}_n]$-modules. The really nice thing about $K[S_n]$-modules or $K[\text{GL}_n]$-modules is that Maschke's Theorem applies and every finietly generated module over these rings is the direct sum of simple modules. The simple modules over $K[S_n]$ or $K[\text{GL}_n]$ were identified by Young and are described using Young Tableau, which are boxes arranged in a stack, corresponding to a partition of n, and filled in with numbers according to some rules. The numbers are usually strictly ascending in one direction and weakly ascending in the other direction.

We return to the regularly scheduled material (before we started thinking about Division rings). We had just defined left ideals, right ideals, and ideals in a ring.

Examples 3.12.

(a) If R is $\mathbb{Z}$ or $F[x]$, where F is a field, then every ideal is principal.[34] Of course, the zero ideal is principal. If I is a non-zero ideal, then let n be the smallest positive element of $\mathbb{Z}$ in I (or f be a non-zero element of I of least degree). If m is an arbitrary element of I, then $m = qn + r$ for integers q and r with $0 \le r \le n-1$. (If $g \in I$, then $g = qf + r$ for polynomials q and r in $F[x]$, where $\deg r < \deg f$.) The fact that r is in I, in each case, forces r to be zero. Thus, $I = (n)$ or $I = (f)$.

(b) The ideals (x, y) of $F[x, y]$, where F is a field and $(2, x)$ of $\mathbb{Z}[x]$ are not principal.

Proof. We focus on the ideal in $F[x, y]$. One can modify our argument to deal with the ideal in $\mathbb{Z}[x]$. First of all, notice that the units of $F[x, y]$ are the non-zero elements of F. (Recall that the element r in the commutative ring R is a unit if there is an element r' in R with $rr' = 1$.) Indeed, if $1 = (\sum_{i=0}^a f_i(x)y^i)(\sum_{j=0}^b g_j(x)y^j)$, with f_a and g_b non-zero, then $a = b = 0$, etc.) Observe that x and y are irreducible[35] elements if $F[x, y]$. The argument starts the same way, if

$$x = \left(\sum_{i=0}^{a} f_i(x)y^i\right)\left(\sum_{j=0}^{b} g_j(x)y^j\right),$$

[34] An ideal of the commutative ring R of the form $\{r_0 r \mid r \in R\}$ for any fixed r_0 in R is called principal and is denoted by r_0R or (r_0). Similarly, if T is any set of elements of the ring R, then (T) or $(T)R$ is the smallest ideal of R which contains T.

[35] The non-zero, non-unit element r of the commutative domain R is irreducible if whenever $r = r_1r_2$ with r_1 and r_2 in R, then r_1 or r_2 is a unit in R.

with f_a and g_b non-zero, then $a+b=0$, $x = f_0(x)g_0(x)$. The rest of the calculation takes place in $F[x]$. The constant terms of f_0 and g_0 must multiply to zero and the degrees of f_0 and g_0 must add to one. The rest is easy. etc.

Suppose $(x, y) = (f)$ for some $f \in F[x, y]$. We produce a contradiction. The elements x and y are irreducible in $F[x, y]$ and f divides x. Thus, f is either a unit or a unit times x. A unit times x can not divide y but f divides y; hence, f must be a unit of $F[x, y]$. Even this is impossible because, $(x, y) \subsetneq F[x, y]$. □

(c) Let R be a ring and i be a fixed integer. The set

$$\{M \in \mathrm{Mat}_{n\times n}(R) | \text{ every entry of column } i \text{ of } M \text{ is zero}\}$$

is I is a left ideal of $\mathrm{Mat}_{n\times n}(R)$. The set

$$\{M \in \mathrm{Mat}_{n\times n}(R) | \text{ every entry of row } i \text{ of } M \text{ is zero}\}$$

is a right ideal of $\mathrm{Mat}_{n\times n}(R)$.

(d) The only two-sided ideals of $\mathrm{Mat}_{n\times n}(F)$, where F is a field, are $\{0\}$ and $\mathrm{Mat}_{n\times n}(F)$. (You can see this easily. If M is a non-zero matrix in an ideal I of $\mathrm{Mat}_{n\times n}(F)$, then by multiplying on the left and on the right you can produce a matrix with exactly one non-zero entry and that entry is 1. Then you can produce matrices in I with 1 in position (i, j) and zero everywhere else for all (i, j), for all (i, j). Then you can conclude $I = \mathrm{Mat}_{n\times n}(F)$.

Observation 3.13. *If I is a two-sided ideal of the ring R, then $\frac{R}{I}$ is a ring with multiplication $\bar{r}\bar{s} = \overline{rs}$ for $r, s \in R$.*

Proof. The quotient $\frac{R}{I}$ is automatically an Abelian group. It is necessary to check that the multiplication is well-defined. If r, r_1, s, s_1 are in R with $\bar{r} = \bar{r}_1$ and $\bar{s} = \bar{s}_1$ in $\frac{R}{I}$, then $r_1 = r + i_1$ and $s_1 = s_i + i_2$ for i_1 and i_2 in I. It follows that

$$r_1 s_1 = (r + i_1)(s + i_2) = rs + \underbrace{i_1 s + r i_2 + i_1 i_2}_{\in I};$$

hence, $\overline{r_1 s_1} = \overline{rs}$ in $\frac{R}{I}$. □

Definition 3.14. If R and S are rings, then the function $\phi : R \to S$ is a ring homomorphism if

$$\begin{aligned}\phi(r + r') =& \phi(r) + \phi(r')\\ \phi(1) =& 1\\ \phi(rr') =& \phi(r)\phi(r').\end{aligned}$$

Theorem 3.15. [First Isomorphism Theorem] *Let $\phi : R \to S$ be a ring homomorphism. Then the following statements hold.*

(a) *The kernel of ϕ is an ideal of R.*

(b) *If I is an ideal of R with $I \subseteq \ker \phi$, then ϕ induces a ring homomorphism $\bar{\phi} : \frac{R}{I} \to S$ with $\bar{\phi}(\bar{r}) = \phi r$.*

(c) *The induced homomorphism $\bar{\phi} : \frac{R}{\ker \phi} \to \operatorname{im} \phi$ is a ring isomorphism.*

Proof. (a) The kernel of ϕ is an Abelian group. We check that $\ker\phi$ is closed under scalar multiplication. If $x \in \ker\phi$ and $r \in R$, then $\phi(rx) = \phi(r)\phi(x) = \phi(r)0 = 0$.

(b) and (c) We verify that $\bar{\phi}$ is a function. If r and r_1 are in R with $\bar{r} = \bar{r}_1$ in $\frac{R}{I}$, then $r - r_1 \in I \subseteq \ker\phi$ and

$$\phi(r) - \phi(r_1) = \phi(r - r_1) = 0.$$

Everything else is automatic. □

Examples 3.16.

(a) The rings $\frac{\mathbb{Z}[x]}{(x^2+1)}$ and $\mathbb{Z}[i]$ are isomorphic.

Proof. Define the ring homomorphism $\phi : \mathbb{Z}[x] \to \mathbb{Z}[i]$ by $\phi(g(x)) = g(i)$. Observe that ϕ is surjective and x^2+1 is in $\ker\phi$. Observe further, that if $f(x)$ is in $\ker\phi$, then $f(x) = q(x^2+1)+r$ where q and r are polynomials in $\mathbb{Z}[x]$ and $r = ax + b$ for some integers a and b.[36] It follows that r is also in $\ker\phi$. The number i is not a rational number; consequently, $ai + b = 0$, with $a, b \in \mathbb{Z}$ only if $a = b = 0$. Thus, $r = 0$ and every element of $\ker\phi$ is in the ideal $(x^2 + 1)$ of $\mathbb{Z}[x]$. Apply the First Isomorphism Theorem to conclude that ϕ induces an isomorphism $\bar{\phi} : \frac{\mathbb{Z}[x]}{(x^2+1)} \to \mathbb{Z}[i]$. □

(b) Let α be a complex number which is algebraic over $\mathbb{Q}$.[37] Let $f(x) \in \mathbb{Q}[x]$ be a non-zero polynomial of least degree with $f(\alpha) = 0$. Define the ring homomorphism

$$\phi : \mathbb{Q}[x] \to \mathbb{Q}[\alpha]$$

by $\phi(g(x)) = g(\alpha)$. Observe that $\ker\phi = (f)$. (Indeed, if $g \in \ker\phi$, then $g = qf + r$ where $q, r \in \mathbb{Q}[x]$ and $\deg r < \deg f$. One sees that $r \in \ker\phi$. The choice of f forces r to be zero.) Conclude that

$$\frac{\mathbb{Q}[x]}{(f)} \cong \mathbb{Q}[\alpha].$$

Definition 3.17. Let R be a commutative ring.

(1) The proper ideal I of R is a maximal ideal if whenever J is an ideal of R with $I \subseteq J \subseteq R$, then $J = I$ or $J = R$.

(2) The proper ideal I of R is a prime ideal if whenever r_1 and r_2 are elements of R, with $r_1 r_2 \in I$, then $r_1 \in I$ or $r_2 \in I$.

Proposition 3.18. *Let I be an ideal of the commutative ring R. Then the following statements hold*:

(a) *I is a prime ideal if and only if R/I is a domain,*

(b) *I is a maximal ideal if and only if R/I is a field, and*

(c) *if I is a maximal ideal, then I is a prime ideal.*

[36] We can use the division algorithm at this point because $x^2 + 1$ is a monic polynomial. A monic polynomial is a polynomial whose leading coefficient is 1.

[37] This means that there exists some non-zero polynomial with coefficients from $\mathbb{Q}$ that α satisfies.

Proof. (a) ($\Leftarrow$) Suppose I is a prime ideal of R. Let r_1, r_2 be elements of R with $\bar{r}_1\bar{r}_2 = \bar{0}$ in $\bar{R} = R/I$. Of course, $\bar{r}_1\bar{r}_2 = \bar{0}$ means $r_1r_2 \in I$. The ideal I is prime; so $r_1 \in I$ or $r_2 \in I$; thus, $\bar{r}_1 = \bar{0}$ or $\bar{r}_2 = \bar{0}$ in $\bar{R}$.

(a) ($\Rightarrow$) Suppose $\bar{R} = R/I$ is a domain. Let r_1, r_2 be elements of R with $r_1r_2 \in I$. It follows that $\bar{r}_1$ and $\bar{r}_2$ are elements of $\bar{R}$ with $\overline{r_1r_2}$, which is equal to $\bar{r}_1\bar{r}_2$, equal to $\bar{0}$. But $\bar{R}$ is a domain; so, $\bar{r}_1 = \bar{0}$ or $\bar{r}_2 = \bar{0}$. In other words, $r_1 \in I$ or $r_2 \in I$.

(b) ($\Leftarrow$) Suppose I is a maximal ideal of R. Let r be an element of R with $\bar{r} \neq \bar{0}$ in $\bar{R} = R/I$. In particular $r \notin I$ and (I, r) is an ideal of R which properly contains the maximal ideal I. It follows that $(I, r) = R$ and there exists $r' \in R$ and $i \in I$ with $i + rr' = 1$ and $\bar{r}\overline{r'} = \bar{1}$.

(b) ($\Rightarrow$) Suppose $\bar{R} = R/I$ is a field. Let r be an element of $R \setminus I$. We show that $(I, r) = R$. The fact that $r \notin I$ ensures that $\bar{r}$ is a unit in $\bar{R}$ so there exists r' in R with $\bar{r}\overline{r'} = \bar{1}$ in $\bar{R}$. In other words, $rr' - 1 \in I$. Thus $1 \in (r, I)$.

(c) If I is a maximal ideal, then R/I is a field. Every field is a domain. Thus, R/I is a domain and therefore I is a prime ideal. □

Examples 3.19.

(a) The ideal (0) is a prime ideal of $\mathbb{Z}$ which is not a maximal ideal. Let n be a non-zero integer. Observe that the following statements are equivalent:

(i) the ideal (n) of $\mathbb{Z}$ is a prime ideal,

(ii) the integer n is irreducible,[38]

(iii) the ideal (n) is a maximal ideal.

Proof.

(ai) $\Rightarrow$ (aii) We prove that if n is a reducible integer, then (n) is not a prime ideal. Of course, this is clear. If $n = n_1n_2$ with n_1, n_2 non-unit integers, then $n_1n_2 \in (n)$ with neither n_1 nor n_2 in (n); thus (n) is not a prime ideal.

(aii) $\Rightarrow$ (aiii) Suppose n is an irreducible integer. We prove that (n) is a maximal ideal. Let J be an ideal of $\mathbb{Z}$ with $(n) \subsetneq J$. The ideal J is principal; so $J = (r)$ for some integer r and $r \notin (n)$. On the other hand $n \in (r)$; so $n = rr'$ for some $r' \in \mathbb{Z}$. The integer n is irreducible; hence either r or r' is a unit times n. We have set things up so that r is not a unit times n; thus, r' is a unit times n and r is in fact a unit. At any rate, (n) is a maximal ideal.

(aiii) $\Rightarrow$ (aii) Apply Proposition 3.18.(c) □

(b) Let $R = F[x]$ be a polynomial ring in one variable over the field F. The ideal (0) is a prime ideal of R which is not a maximal ideal. Let f be a non-zero polynomial in R. Observe that the following statements are equivalent:

(i) the ideal (f) of $F[x]$ is a prime ideal,

[38] Recall that we proved a little unit about the factorization of integers which starts with Definition 2.32.

(ii) the polynomial f is irreducible,[39]
(iii) the ideal (f) is a maximal ideal.

Proof. Use the same proof as was given in (a). □

(c) Observe that $(0) \subsetneq (2) \subsetneq (2, x)$ is a chain of prime ideals in $\mathbb{Z}[x]$. In this chain, only $(2, x)$ is a maximal ideal.
(d) Observe that

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq (x_1, x_2, x_3, x_4) \subsetneq (x_1, x_2, x_3, x_4, x_5)$$

is a chain of prime ideals in the polynomial $F[x_1, x_2, x_3, x_4, x_5]$ over the field F. The only maximal ideal in this chain is $(x_1, x_2, x_3, x_4, x_5)$.

Corollary 3.20. *If $\alpha \in \mathbb{C}$ and α is algebraic over $\mathbb{Q}$, then $\mathbb{Q}[\alpha]$ is a field.*

Proof. We saw in Example 3.16.b that

$$\mathbb{Q}[\alpha] \cong \frac{\mathbb{Q}[x]}{(f)},$$

where f is a non-zero polynomial of $\mathbb{Q}[x]$ of least degree with $f(\alpha) = 0$. The ring $\mathbb{Q}[\alpha]$ is a subring of $\mathbb{C}$; so $\mathbb{Q}[\alpha]$ is a domain. Thus, (f) is a non-zero prime ideal of $\mathbb{Q}[x]$. Every non-zero prime ideal of $\mathbb{Q}[x]$ is a maximal ideal, by Example 3.19.(b). Thus, (f) is a maximal ideal of $\mathbb{Q}[x]$ and $\mathbb{Q}[x]/(f)$ is a field by Proposition 3.18. □

Corollary 3.21. *Let $\alpha, \beta \in \mathbb{C}$ with α and β algebraic over $\mathbb{Q}$. Let $f(x) \in \mathbb{Q}[x]$ be the minimal polynomial of α over $\mathbb{Q}$. (Thus, $f \in \mathbb{Q}[x]$ is a non-zero polynomial of least degree with $f(\alpha) = 0$.) Suppose $f(\beta) = 0$. Then the rings $\mathbb{Q}[\alpha]$ and $\mathbb{Q}[\beta]$ are isomorphic.*

Proof. Apply Example 3.16.(b) to see that $\mathbb{Q}[\alpha] \cong \frac{\mathbb{Q}[x]}{(f)}$. The ring $\mathbb{Q}[\alpha]$ is a domain; so (f) is a prime ideal and f is an irreducible polynomial. Let $g \in Q[x]$ be the minimal polynomial of β. It follows that $(g) = \{h \in \mathbb{Q}[x] \mid h(\beta) = 0\}$. The hypothesis that $f(\beta) = 0$ guarantees that $f \in (g)$ and $f = g'g$ for some $g' \in \mathbb{Q}[x]$. The polynomial f is irreducible and g is not a unit; hence, f is a unit times g and f is also a minimal polynomial of β. Thus, $\mathbb{Q}[\beta] \cong \frac{\mathbb{Q}[x]}{(f)}$ and the proof is complete. □

Observation 3.22. [The second isomorphism Theorem] *If I is an ideal of the ring R, then the following statements hold.*

(a) *Every ideal of $\frac{R}{I}$ has the form $\frac{J}{I} = \{\bar{j} \mid j \in J\}$, where J is an ideal of R with $I \subseteq J$.*
(b) *There exists a one-to-one correspondence between the ideals of R which contain I and the ideals of $\frac{R}{I}$.*

[39] The factorization of polynomials in $F[x]$ works just like the factorization in $\mathbb{Z}$. Re-write the little factorization unit which starts with Definition 2.32 to be about ideals in a Principal Ideal Domain rather than about subgroups of $\mathbb{Z}$. (Of course a Principal Ideal Domain, is a domain in which every ideal is principal.) Recall from Example 3.12.(a) that $F[x]$ is a Principal Ideal Domain.

(c) *If J is an ideal of R which contains I, then*

$$\frac{\frac{R}{I}}{\frac{J}{I}} \cong \frac{R}{J}.$$

Proof. The arguments for (a) and (b) are straightforward. We prove (c). Apply the First Isomorphism Theorem to the natural quotient map

$$\phi : R \to R/J.$$

Observe that $I \subseteq \ker \phi$, so ϕ induces a ring homomorphism $\bar{\phi} : R/I \to R/J$ with $\bar{\phi}(\bar{r}) = \phi(r) = \bar{r}$. Of course, $\bar{r}$ on the left is an element of R/I and $\bar{r}$ on the right is an element of R/J. Apply the First Isomorphism Theorem again to the ring homomorphism $\bar{\phi} : R/I \to R/J$ to see that

$$\frac{R/I}{\ker \bar{\phi}} \cong R/J.$$

There is no difficulty in computing that $\ker \bar{\phi} = J/I$. □

3.C. **The quotient field of a domain.** The basic thought is that every domain sits inside a field. Indeed, every domain sits inside a "smallest field". There is no well-defined ordering on the set of fields which contain a given domain; so we can not just intersect over all of the fields containing the domain. Instead, we use a Universal Mapping Property.

Definition 3.23. Let D be a domain, F be a field, and $i : D \to F$ be an injective ring homomorphism. Then F is the quotient field of D (and $i : D \to F$ is a quotient field map) if $i : D \to F$ satisfies the following Universal Mapping Property: Whenever $\Phi : D \to K$ is an injective ring homomorphism to a field, then there exists a unique ring homomorphism $\phi : F \to K$, so that the following diagram commutes

$$\begin{tikzcd}
D \arrow[rr, hook, "i"] \arrow[dr, hook, "\Phi"'] & & F \arrow[dl, dashed, "\exists!\phi"] \\
& K. &
\end{tikzcd}$$

Observation 3.24. *Let D be a domain. If D has a quotient field, then this quotient field is unique.*

Proof. Suppose that $i_j : D \to F_j$ both satisfy the UMP for quotient field, for $1 \le j \le 2$.

$$\begin{tikzcd}
D \arrow[rr, hook, "i_1"] \arrow[dr, "i_2"] \arrow[dd, hook, "i_1"'] & & F_1 \arrow[dl, dashed, "\exists!\phi_1"] \\
& F_2 \arrow[dl, dashed, "\exists!\phi_2"] & \\
F_1. & &
\end{tikzcd}$$

The hypothesis that $i_1 : D \to F_1$ is a quotient field map ensures that there exists a unique ring homomorphism $\phi_1 : F_1 \to F_2$ such that $\phi_1 \circ i_1 = i_2$. Similarly, the hypothesis that $i_2 : D \to F_2$ is a quotient field map ensures that there exists a unique ring homomorphism $\phi_2 : F_2 \to F_1$ such that $\phi_2 \circ i_2 = i_1$. We now have two maps $F_1 \to F_1$ which cause

$$\begin{tikzcd}
D \arrow[r, hook, "i_1"] \arrow[d, hook, "i_1"'] & F_1 \arrow[dl, dashed, "\exists!"] \\
F_1 &
\end{tikzcd}$$

to commute; namely id_{F_1} and $\phi_2 \circ \phi_1$. We conclude that $\phi_2 \circ \phi_1 = \mathrm{id}_{F_1}$. Thus, ϕ_1 is injective and ϕ_2 is surjective. Reverse the roles of F_1 and F_2 to conclude that ϕ_2 is injective and ϕ_2 is surjective. We conclude that $\phi_1 : F_1 \to F_2$ is an isomorphism and

$$\begin{tikzcd}
D \arrow[r, hook, "i_1"] \arrow[d, hook, "i_2"'] & F_1 \arrow[dl, "\phi_1"] \\
F_2 &
\end{tikzcd}$$

commutes. □

Example 3.25. If F is a field then the identity map $\mathrm{id} : F \to F$ is a quotient field map.

Proof. If $\Phi : F \to K$ is any injective ring homomorphism into a field, then there does indeed exist a unique ring homomorphism $F \to K$ such that the diagram

$$\begin{tikzcd}
F \arrow[r, hook, "\mathrm{id}"] \arrow[d, hook, "\Phi"'] & F \arrow[dl, dashed, "\exists!"] \\
K &
\end{tikzcd}$$

commutes; namely Φ. □

Observation 3.26. *The inclusion map $i : \mathbb{Z} \to \mathbb{Q}$ is a quotient field map.*

Proof. Suppose $\mathbb{Z} \xhookrightarrow{\Phi} K$ is an injection into a field. We must prove that there exists a unique ring homomorphism $\phi : \mathbb{Q} \to K$ such that $\phi \circ i = \Phi$.

We first show that there is only one candidate for ϕ. Let b be a non-zero integer. If $\phi : \mathbb{Q} \to K$ is a ring homomorphism with $\phi \circ i = \Phi$, then

$$\phi(\tfrac{a}{b}) = \phi(a)\phi(\tfrac{1}{b}) = (\phi \circ i)(a)\phi(\tfrac{1}{b}) = \Phi(a)\phi(\tfrac{1}{b}).$$

Furthermore,

$$1 = \Phi(1) = \phi(i(1)) = \phi(1) = \phi(\tfrac{b}{b}) = \phi(b)\phi(\tfrac{1}{b}) = (\phi \circ i)(b)\phi(\tfrac{1}{b}) = \Phi(b)\phi(\tfrac{1}{b}).$$

The hypothesis ensures that $\Phi(b) \neq 0$. Thus $\Phi(b)$ has an inverse in K. We conclude that

$$(\Phi(b))^{-1} = \phi(\tfrac{1}{b}) \quad \text{and} \quad \phi(\tfrac{a}{b}) = \Phi(a)(\Phi(b))^{-1}.$$

Now we show that $\phi : \mathbb{Q} \to K$, with $\phi(\frac{a}{b}) = \Phi(a)(\Phi(b))^{-1}$, for $a, b \in \mathbb{Z}$ and $b \neq 0$, is a function.

Suppose a, b, c, d are integers with $b \neq 0$, $d \neq 0$, and $\frac{a}{b} = \frac{c}{d}$ in $\mathbb{Q}$. We show that

$$\Phi(a)(\Phi(b))^{-1} = \Phi(c)(\Phi(d))^{-1}$$

in K. Well,

$$ad = bc$$

in $\mathbb{Q}$ and also in $\mathbb{Z}$; thus

$$\Phi(a)\Phi(d) = \Phi(ad) = \Phi(bc) = \Phi(b)\Phi(c). \tag{3.26.1}$$

The homomorphism Φ is injective and b and d are non-zero integers; hence, $\Phi(b)$ and $\Phi(d)$ are non-zero elements of the field K. In particular, $\Phi(b)$ and $\Phi(d)$ have inverses in K. Multiply both sides of (3.26.1) by $\Phi(b)^{-1}\Phi(d)^{-1}$ and use the hypothesis that multiplication in K commutes to see that

$$\Phi(a)(\Phi(b))^{-1} = \Phi(c)(\Phi(d))^{-1}$$

as desired.

Now we show that the function $\phi : \mathbb{Q} \to K$, with $\phi(\frac{a}{b}) = \Phi(a)(\Phi(b))^{-1}$, for $a, b \in \mathbb{Z}$ and $b \neq 0$, is a ring homomorphism.

If a, b, c, d are integers with b and d not zero, then

$$\phi(\tfrac{a}{b} + \tfrac{c}{d}) = \phi(\tfrac{ad+bc}{cd}) = \Phi(ad + bc)(\Phi(bd))^{-1} = \Phi(a)(\Phi(b))^{-1} + \Phi(c))(\Phi(d))^{-1} = \phi(\tfrac{a}{b}) + \phi(\tfrac{c}{d}),$$

$$\phi(\tfrac{a}{b}\tfrac{c}{d}) = \phi(\tfrac{ac}{bd}) = \Phi(ac)(\Phi(bd))^{-1} = \phi(\tfrac{a}{b})\phi(\tfrac{c}{d}),$$

and

$$\phi(1) = \Phi(1) = 1.$$

Finally, we record the fact that the function $\phi : \mathbb{Q} \to K$, with $\phi(\frac{a}{b}) = \Phi(a)(\Phi(b))^{-1}$, for $a, b \in \mathbb{Z}$ and $b \neq 0$, satisfies $\phi \circ i = \Phi$.

Of course, this is clear. Indeed, if $n \in \mathbb{Z}$, then

$$(\phi \circ i)(n) = \phi(\tfrac{n}{1}) = \Phi(n)(\Phi(1))^{-1} = \Phi(n).$$

□

Proposition 3.27. *If D is a domain, then there exists a field F and a quotient field map $D \xhookrightarrow{i} F$.*

Proof. Consider the set

$$S = \{(a, b) \mid a, b \in D \text{ with } b \neq 0\}.$$

Consider the relation $\sim$ on S where if (a, b) and (c, d) are in S, then

$$(a, b) \sim (c, d) \iff ad = bc \in D.$$

Observe that $\sim$ is an equivalence relation. Let F be the set of equivalence classes $S/\sim$. In other words, the elements of F all have the form $\overline{(a, b)}$, where (a, b) is an element of S and if (a, b) and (c, d) are elements of S, then $\overline{(a, b)} = \overline{(c, d)}$ in F if and only if $(a, b) \sim (c, d)$. Observe that the following statements hold.

(1) The function $S \times S \to S$, which is given by $(a, b)+(c, d) = (ac+bd, bd)$, induces a well-defined function $F \times F \to F$.

(2) The function $S \times S \to S$, which is given by $(a, b) \cdot (c, d) = (ac, bd)$. induces a well-defined function $F \times F \to F$.

(3) The set F with operations $+$ and $\cdot$, described in (1) and (2), forms a field. The additive identity of F is $\overline{(0,1)}$; the multiplicative identity is $\overline{(1,1)}$.
(4) The function $i : D \to F$, given by $d \mapsto \overline{(d,1)}$, is a ring homomorphism.
(5) The ring homomorphism i of (4) satisfies the Universal Mapping Property of a quotient field homomorphism.

□

3.D. **Math 702 starts here.**

- I am using the website:
  `https://people.math.sc.edu/kustin/teaching/702/702.html`
- I posted HW1, due on Jan 29.
- The syllabus has the exam dates.
- Are there comments or question?

In December we defined the quotient field of a domain.

Definition. 3.23 Let D be a domain, F be a field, and $i : D \to F$ be an injective ring homomorphism. Then F is the quotient field of D (and $i : D \to F$ is a quotient field map) if $i : D \to F$ satisfies the following Universal Mapping Property: Whenever $\Phi : D \to K$ is an injective ring homomorphism to a field, then there exists a unique ring homomorphism $\phi : F \to K$, so that the following diagram commutes

$$\begin{tikzcd}
D \arrow[r, hook, "i"] \arrow[dr, hook, "\Phi"'] & F \arrow[d, dashed, "\exists!\phi"] \\
& K.
\end{tikzcd}$$

We proved that if the domain D has a quotient field, then that quotient field is unique. We started to prove that $\mathbb{Q}$ is the quotient field of $\mathbb{Z}$. (I am not going to do more of this in public, but you are welcome to look at the typed notes.)

Proposition 3.28. *If D is a domain, then there exists a field F and a quotient field map* $D \xhookrightarrow{i} F$.

Proof. Consider the set

$$S = \{(a, b) \mid a, b \in D \text{ with } b \neq 0\}.$$

Consider the relation $\sim$ on S where if (a, b) and (c, d) are in S, then

$$(a, b) \sim (c, d) \iff ad = bc \in D.$$

Observe that $\sim$ is an equivalence relation. Let F be the set of equivalence classes $S/\sim$. In other words, the elements of F all have the form $\overline{(a, b)}$, where (a, b) is an element of S and if (a, b) and (c, d) are elements of S, then $\overline{(a, b)} = \overline{(c, d)}$ in F if and only if $(a, b) \sim (c, d)$. Observe that the following statements hold.

(1) The function $S \times S \to S$, which is given by $(a, b)+(c, d) = (ac+bd, bd)$, induces a well-defined function $F \times F \to F$.
(2) The function $S \times S \to S$, which is given by $(a, b) \cdot (c, d) = (ac, bd)$. induces a well-defined function $F \times F \to F$.
(3) The set F with operations $+$ and $\cdot$, described in (1) and (2), forms a field. The additive identity of F is $\overline{(0, 1)}$; the multiplicative identity is $\overline{(1, 1)}$.
(4) The function $i : D \to F$, given by $d \mapsto \overline{(d, 1)}$, is a ring homomorphism.
(5) The ring homomorphism i of (4) satisfies the Universal Mapping Property of a quotient field homomorphism.

□

3.E. **Unique Factorization Domains.**

Definition 3.29. The domain D is a Unique Factorization Domain (UFD) if

(a) Every non-zero element of D which is not a unit is a finite product of irreducible elements.
(b) If $d = \prod_{i=1}^{r} p_i$ and $d = \prod_{i=1}^{s} q_i$ are two factorizations of the non-zero non-unit d into irreducible elements, then $r = s$ and after renumbering $p_i = \text{unit}_i q_i$ for all i.

Examples 3.30. (a) Every field is a UFD. (Indeed, every element of a field is zero or a unit.)
(b) Every PID is a UFD. (This is a Theorem. We have essentially proved it. We will tidy it up.) Just in case I neglected to define PID, I include: The domain R is a Principal Ideal Domain (PID) if every ideal of R has the form (r) for some r in R. The standard examples of PIDs are: every field is a PID; the ring of integers is a PID; if F is a field, then the polynomial ring $F[x]$ (in one variable!) is a PID.)
(c) If D is a UFD, then $D[x]$ is a UFD. (This is a Theorem. Essentially, it is due to Gauss.) In particular, if D is a PID (or a field), then $D[x_1, \dots, x_n]$ is a UFD.
(d) If P is a smooth point on an Algebraic variety X, then the ring of rational functions on X which are defined at P (usually denoted $\mathscr{O}_{X,P}$) is a UFD. This theorem is due to Auslander-Buchsbaum-Serre 1959. This Theorem made Algebraic Geometers pay attention to Homological Algebra.
(e) The ring $\mathbb{Z}[\sqrt{-5}]$ is not a UFD. (This is homework problem 1.)
(f) The ring $R = F[x, y, z, w]/(xy - zw)$ is not a UFD, where F is a field. The elements x, y, z, w of R are all irreducible and none is a unit times another. But $xy = zw$ in R. If you care, R is the homogeneous coordinate ring of the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$. I can unpack that. Projective one space is

$$\frac{\{[x_0 : x_1] \mid (x_0, x_1) \in F^2 \setminus \{(0,0)\}\}}{[x_0 : x_1] \sim [tx_0 : tx_1] \text{ for } t \in F \setminus \{0\}}$$

So, $\mathbb{P}^1$ is ordinary affine one space $\{[1 : x_1]\}$ together with a point at infinity $[0 : 1]$. Projective space is nice because it is compact (in the Zariski topology) and all the points look alike. If you happen to be standing at $[0 : 1]$, then from your point of view $[1 : 0]$ is "infinity". One disadvantage to projective space is that the product of projective spaces is not a projective space. Ah, but one can embed a product of projective spaces in projective space. The Segre embedding of

$$\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$

is

$$([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1].$$

(Be sure to notice that this function is well-defined.) If the coordinates of $\mathbb{P}^3$ are $[x : z : w : y]$, then the set of polynomials that vanish on the image of the above Segre embedding is the ideal $(xy - zw)$ in the polynomial ring $F[x, y, z, w]$. (One direction of this assertion is obvious. The other direction requires a little work, but not much.) Hence the homogeneous coordinate ring for the image of this Segre embedding is

$$\frac{F[x,y,z,w]}{(xy-zw)}.$$

(g) The ring $\mathbb{Z}[i]$ is a Euclidean Domain; hence a PID; hence a UFD. This is homework problem 4.

History 3.31. Fermat's Last Theorem was conjectured in 1640. It states that $x^n + y^n = z^n$ has no positive integer solutions for $3 \leq n$.

In the mid-1800's it was realized that in $\mathbb{Z}[\omega_p]$ every non-zero element which is not a unit can be factored into irreducible elements, where $\omega_p = e^{2\pi i/p}$. (In this discussion, p is a prime integer.) It was assumed that this factorization is unique. A famous false proof of FLT was given under the hypothesis that $\mathbb{Z}[\omega_p]$ is a UFD for all primes p. Dirichlet observed that $\mathbb{Z}[\omega_{23}]$ is not a UFD. Kummer invented "ideal theory". (He considered "idealized integers". These are the objects we call ideals.) The rings $\mathbb{Z}[\omega_p]$ and $\mathbb{Z}[\sqrt{-5}]$ are examples of "rings of algebraic integers" – each element in these rings satisfies a **monic** polynomial with integer coefficients. Every ring of algebraic integers is a Dedekind Domain. Kummer proved that in a Dedekind domain every ideal can be factored into a product of prime ideals in a unique manner. (Homework problem 2 asks you to factor a particular ideal into a product of two ideals.)

An aside: The fact that the factorization is unique is easy but requires a little more than you know. The argument goes something like this. Suppose $P_1, \dots P_a$ and $Q_1, \dots, Q_b$ are prime ideals in the Dedekind Domain R and

$$\prod P_i = \prod Q_j, \tag{3.31.1}$$

then $a = b$ and, after re-numbering $P_i = Q_i$. The fact that $\prod Q_j \subseteq P_1$ and P_1 is prime forces $Q_j \subseteq P_1$, for some j. (Renumber the Q's to get $Q_1 \subseteq P_1$.) In a Dedekind domain all non-zero primes ideals are maximal ideals. (You probably do not know this.) So $Q_1 = P_1$. The next thing you don't know is that each non-zero ideal I in a Dedekind Domain has an "inverse". This "inverse" is a finitely generated R-submodule I^{-1} of the quotient field K of R with $II^{-1} = R$. (In this context an R-submodule of K is an Abelian group which is a subgroup of $(K, +)$ and is closed under scalar multiplication by R.) Multiply both sides of equation (3.31.1) by P_1^{-1} and repeat or use induction.

Theorem 3.32. *If R is a PID, then R is a UFD.*

We have essentially established this very important Theorem.

Definition 3.33. The ring R is Noetherian if the ideals of R satisfy the Ascending Chain Condition (ACC). The ideals of R satisfy (ACC) if whenever

$$I_1 \subseteq I_2 \subseteq \cdots$$

is an ascending chain of ideals of R, then there exists an integer n such that $I_n = I_{n+k}$ for all non-negative integers k.

Remark 3.34. I am thinking of R as a commutative ring; but it could just as well be non-commutative. If R is non-commutative one could say that R is left-Noetherian if the left ideals of R satisfy (ACC) or right-Noetherian if the right ideals of R satisfy (ACC).

Observation 3.35. *If R is a (commutative) ring, then R is Noetherian if and only if every ideal of R is finitely generated.*

Proof. Suppose every ideal of R is finitely generated and

$$I_1 \subseteq I_2 \subseteq \cdots$$

is an ascending chain of ideals of R. Then $\cup_i I_i$ is an ideal. This ideal is finitely generated and all of the generators are in I_n for some n. It follows that $I_n = I_{n+k}$ for all non-negative k.

Suppose some ideal I is not finitely generated. Take $i_1 \in I$. Then $(i_1) \subsetneq I$. It follows that there is an element $i_2 \in I \setminus (i_1)$. Thus,

$$(i_1) \subsetneq (i_1, i_2) \subsetneq I.$$

Continue in this manner to build an ascending chain of ideals of R which never stabilizes. □

Observation 3.36. *If R is a Noetherian domain, and r is an element of R which is not zero and is not a unit, then r is a finite product of irreducible elements of R.*

Proof. Modify the proof given in Lemmas 2.36 and 2.37. In these results we proved that every integer is a finite product of irreducible integers. The only property about integers that we used is that the subgroups of the group $(\mathbb{Z}, +)$ satisfy the ascending chain condition. Notice that the subgroups of the group $(\mathbb{Z}, +)$ are exactly the same as the ideals of the ring $\mathbb{Z}$. □

Corollary 3.37. *If r is an element of the PID R and r is not zero and not a unit, then r is a finite product of irreducible elements.* □

Proposition 3.38. *If r is an element of the PID R and r is not zero and not a unit, then r is irreducible if and only if (r) is a prime ideal.*

Proof. In fact, the following three statements are equivalent because R is a PID:

(a) r is an irreducible element of R,
(b) (r) is a maximal ideal of R, and
(c) (r) is a prime ideal of R.

To prove this equivalence, modify Example 3.19 as needed. □

Proposition 3.39. *The domain R is a UFD if and only if every non-zero non-unit element of R is a finite product of irreducible elements and every irreducible element of R generates a prime ideal.*

Proof. If necessary, one should look at the proof of Theorem 2.33 on page 24. □

This completes the proof of Theorem 3.32.

I do want to prove the Hilbert Basis Theorem. If R is a Noetherian (commutative) ring, then $R[x]$ is a Noetherian ring. (Hence as a consequence,

$$\frac{R[x_1, \dots, x_n]}{I}$$

is a Noetherian ring for any non-negative integer n and any ideal I of $R[x_1, \dots, x_n]$.) So, you would have to work hard to find a domain R and an element r in R with r not zero, r not a unit, and r not equal to a finite product of irreducible elements. (This is homework problem 3.)

Theorem 3.40. **[The Hilbert Basis Theorem]** *If R is a (commutative) Noetherian ring, then $R[x]$ is also a Noetherian ring.*

Proof. Let J be an ideal of $R[x]$. For each integer n, let

$$I_n = \{r \in R | rx^n + \text{l.o.t.} \in J\}.$$

Observe that

$$I_0 \subseteq I_2 \subseteq \cdots$$

are ideals of R. Every ascending chain of ideals in R stabilizes; hence, there exists n_0 with

$$I_{n_0+k} = I_{n_0}$$

for all non-negative k. For each i with $0 \le i \le n_0$ pick polynomials $f_{i,1}, \dots, f_{i,N_i}$ in $R[x]$ such that each $f_{i,j}$ is in J and has degree i; furthermore the leading coefficients of $f_{i,1}, \dots, f_{i,N_i}$ generate I_i. I claim that

$$\bigcup_{i=0}^{n_0} \{f_{i,1}, \dots, f_{i,N_i}\}$$

generates J. It is clear that

$$\left(\bigcup_{i=0}^{n_0} \{f_{i,1}, \dots, f_{i,N_i}\}\right) \subseteq J.$$

We prove the other inclusion. Let f be an element of J. We prove that

$$f \in \left(\bigcup_{i=0}^{n_0} \{f_{i,1}, \dots, f_{i,N_i}\}\right)$$

by induction on the degree of f. It is clear that if $\deg f = 0$, then $f \in (f_{0,1}, \dots, f_{0,N_0})$. Suppose that $g \in J$ with $\deg g < \deg f$ implies that

$$g \in \left(\bigcup_{i=0}^{n_0} \{f_{i,1}, \dots, f_{i,N_i}\}\right)$$

Observe that f minus a linear combination of elements from

$$\bigcup_{i=0}^{n_0} \{f_{i,1}, \dots, f_{i,N_i}\}$$

is in J and has degree less than the degree of f. (This works for $\deg f \le n_0$ as well as $n_0 < \deg f$.) Hence the proof is complete by induction. □

Theorem 3.41. *If R is a* UFD*, then $R[x]$ is a* UFD.

The idea: The ring $R[x]$ sits between two UFDs:

$$R \subseteq R[x] \subseteq K[x]$$

where K is the quotient field of R. We will prove that the irreducible elements of $R[x]$ are the irreducible elements of R and the elements f of $R[x]$ such that the coefficients of f are relatively prime and f is irreducible in $K[x]$.

Definition 3.42. Let R be a UFD and $f \in R[x]$. If the coefficients of f are relatively prime, then f is called a $\underline{\text{primitive}}$ polynomial.

Lemma 3.43. [Gauss' Lemma] *Let R be a* UFD. *If f and g are primitive polynomials in $R[x]$, then fg is primitive.*

Proof. Let $f = \sum_{i=0}^{s} f_i x^i$ and $g = \sum_{j=0}^{t} g_j x^j$, with f_i and g_j in R. Let p be an arbitrary irreducible element of R. Suppose that $p|f_i$ for $i < a$ and p does not divide f_a and $p|g_j$ for $j < b$ and p does not divide g_b. Observe that the coefficient of x^{a+b} in fg is

$$\underbrace{\cdots + f_{a-1}g_{b+1} +}_{p|\text{ this}} \quad \underbrace{f_a g_b}_{p \text{ does not divide this}} \quad \underbrace{+ f_{a+1}g_{b-1} + \dots}_{p|\text{ this}}.$$

Thus, p does not divide the coefficient of x^{a+b} in fg. We conclude that fg is primitive. □

Corollary 3.44. *Let R be a UFD and f be a primitive polynomial in $R[x]$. Let K be the quotient field of R.*

(a) *Let $f = \prod_{i=1}^{s} g_i$ in $K[x]$ with $g_i = \frac{a_i}{b_i} h_i$ for a_i, b_i in R and h_i a primitive polynomial of $R[x]$. Then f is equal to $\prod_{i=1}^{s} h_i$ times a unit of R.*

(b) *The polynomial f is irreducible in $R[x]$ if and only if f is irreducible in $K[x]$.*

Proof. (a) Observe that $(\prod b_i) f = (\prod a_i)(\prod h_i)$. The polynomials f and $\prod h_i$ are both primitive in $R[x]$. (Use Gauss' Lemma for $\prod h_i$.) Hence $\prod a_i$ is equal to a unit of R times $\prod b_i$ in R.

(b) Suppose f is irreducible in $R[x]$ and $f = g_1 g_2$ in $K[x]$. Write $g_i = \frac{a_i}{b_i} h_i$ with a_i, b_i in R and h_i primitive in $R[x]$. Apply (a) to conclude that f is equal to $h_1 h_2$ times a unit of R in $R[x]$. The polynomial f is irreducible in $R[x]$; so h_1 or h_2 is a unit of $R[x]$; hence a unit of R. Thus, either g_1 or g_2 is a unit in $K[x]$ and f is irreducible in $K[x]$.

Suppose f is irreducible in $K[x]$. Suppose $f = g_1 g_2$ in $R[x]$. The hypothesis that f is irreducible in $K[x]$ ensures that either g_1 or g_2 is a unit of $K[x]$; hence, an element of K. But f is primitive in $R[x]$. Thus, g_1 or g_2 is a unit in R. We conclude that f is irreducible in $R[x]$. □

The proof of Theorem 3.41. Take $f \in R[x]$. Use Corollary 3.44.(a) to write $f = \prod r_i \prod f_j$ where the r_i are irreducible in R and the f_j are primitive in $R[x]$ and irreducible in $K[x]$. We have exhibited a factorization of f into irreducible elements of $R[x]$.

Suppose

$$\prod r_i \prod f_j = \prod s_k \prod g_\ell \tag{3.44.1}$$

with r_i and s_k irreducible in R and f_j and g_ℓ primitive in $R[x]$ and irreducible in $K[x]$. First look in $K[x]$ to see that there are exactly as many f_j's as there are g_ℓ's and after renumbering f_j is equal to a unit of K times g_j. The unit of K is a/b for a, b in R with b not zero. Multiply both sides by b: $bf_j = ag_j$. The polynomials f_j and g_j are primitive in $R[x]$; hence, b is equal to a unit of R times a. At any rate, we can cancel all of the f_j's and g_ℓ's from (3.44.1) at the expense of multiplying one of the sides by a unit of R. We are left with $\prod r_i$ is equal to a unit of R times $\prod s_k$, where the r_i and the s_k are irreducible elements of R. The hypothesis that R is a UFD ensures that the number of r_i is equal to the number of s_k and after re-numbering r_i is equal to a unit of R times s_i. □

4. Modules

Definition 4.1. Let R be a (commutative) ring. An R-module is an Abelian group M with a scalar multiplication which satisfies

(a) If $r \in R$ and $m \in M$, then $rm \in M$.
(b) If r, s are in R and $m \in M$, then $r(sm) = (rs)m$.
(c) If $r \in R$ and $m, m' \in M$, then $r(m + m') = rm + rm'$.
(d) If $r, r' \in R$ and $m \in M$, then $(r + r')m = rm + r'm$.
(e) If 1 is the multiplicative identity element of R and $m \in M$, then $1m = m$.

If M and M' are R-modules, then an R-module homomorphism $f : M \to M'$ is a homomorphism of Abelian groups which satisfies

$$f(rm) = rf(m)$$

for all $r \in R$ and $m \in M$.

Examples 4.2. (a) Every Abelian group is a $\mathbb{Z}$-module. Every homomorphism of Abelian groups is a $\mathbb{Z}$-module homomorphism.
(b) Every vector space over the field $\boldsymbol{k}$ is a $\boldsymbol{k}$-module. Every linear transformation of vector spaces over $\boldsymbol{k}$ is a $\boldsymbol{k}$-module homomorphism.
(c) The ring R is a module over itself.
(d) If I is an ideal of the ring R, then I is an R-module and the inclusion map $I \hookrightarrow R$ is an R-module homomorphism.
(e) If $N \subseteq M$ are R-modules, then the Abelian group M/N is an R-module.
(f) Let I be an index set. If M_i is an R-module for all i in I, then the Abelian groups $\prod_{i \in I} M_i$ and $\bigoplus_{i \in I} M_i$ are R-modules. The R-module $\bigoplus_I R$ is called a free R-module. It is easy to create R-modules homomorphisms from a free R-module, you are free to send each basis element wherever you want. For example, if M is any module and $m_1, \dots, m_n$ are arbitrary elements of M, then there is an R-module homomorphism $\phi : \bigoplus_{i=1}^n R \to M$ with

$$\phi\left(\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}\right) = \sum_{i=1}^{n} r_i m_i.$$

(g) If $f : M \to N$ is an R-module homomorphism, then the kernel, image, and cokernel of f are all R-modules.
(h) If V is a vector space over a field $\boldsymbol{k}$ and $T : V \to V$ is a linear transformation of $\boldsymbol{k}$-vector spaces, then V is a $\boldsymbol{k}[x]$-module with $xv = T(v)$.
(i) If $\phi : R \to S$ is a ring homomorphism, then S is an R-module with $rs = \phi(r)s$ for $r \in R$ and $s \in S$, where rs means scalar multiplication and $\phi(r)s$ means multiplication in the ring S.
(j) Every module is the cokernel of a homomorphism between two free modules. Let R be a (commutative) ring and M be an R-module. Pick a generating set $\{m_i | i \in I\}$ for M. Define a surjective R-module homomorphism $\pi : \bigoplus_{i \in I} R \twoheadrightarrow M$. Pick a generating set $\{\theta_j \mid j \in J\}$ for $\ker \pi$. Define an R-module $\phi : \bigoplus_{j \in J} R \twoheadrightarrow \ker \pi$. Let $\Phi : \bigoplus_{j \in J} R \to \bigoplus_{i \in I} R$ be the

composition

$$\bigoplus_{j\in J} R \xrightarrow{\phi} \ker \pi \xrightarrow{\text{inclusion}} \bigoplus_{i\in I} R.$$

Observe that M is isomorphic to the cokernel of Φ. Indeed,

$$\operatorname{coker} \Phi = \frac{\bigoplus_{i\in I} R}{\operatorname{im} \Phi} = \frac{\bigoplus_{i\in I} R}{\ker \pi} \cong M.$$

The left most equality is the definition of cokernel. The next equality is due to the definition of Φ. The isomorphism is the First Isomorphism Theorem. In particular, if R is a Noetherian ring and M is a finitely generated R-module, then the index sets can be chosen to be finite. (This is a homework problem!) In this case, Φ is multiplication by a matrix with entries in R. When I want the computer to consider a module, I give it the presentation matrix Φ.

Theorem 4.3. *If R is a Euclidean Domain and M is a finitely generated R-module, then the following statements hold.*

(a) *The R-module M is a direct sum of cyclic R-modules.*

(b) *There is a non-negative integer a and there are elements $f_1, \cdots, f_s$ in R such that*

$$M \cong R^a \oplus \frac{R}{(f_1)} \oplus \cdots \oplus \frac{R}{(f_s)}$$

and $f_1 | f_2 | \cdots | f_s$. Furthermore, the integer a and the ideals $(f_1), \dots, (f_s)$ of R are uniquely determined by M.

(c) *There is a non-negative integer a, there are prime ideals $(p_1), \dots, (p_t)$, and there are positive integers $e_1, \dots, e_t$ such that*

$$M \cong R^a \oplus \bigoplus_{i=1}^{t} \frac{R}{(p_i^{e_i})}.$$

Furthermore, the integer a is unique and the ideals $(p_1), \dots, (p_t)$ of R and the exponents $e_1, \dots, e_t$ are uniquely determined by M up to re-ordering.

Proof. Assertions (b) and (c) follow[40] from (a) and the Chinese Remainder Theorem[41] (for PIDs) exactly as in the case of finite Abelian groups. The proof of (a) for Euclidean Domains is exactly the same as the proof of the corresponding result for Abelian groups; see Lemma 2.94. □

Remark 4.4. Theorem 4.3 holds over any PID. But the argument for (a) for arbitrary PIDs rather than only Euclidean Domains requires more careful calculations. **I do not plan to do this more careful proof.**

4.A. **The canonical forms of matrices.**

Definition 4.5. Let $\boldsymbol{k}$ be a field and M and N be $n \times n$ matrices with entries in $\boldsymbol{k}$. The matrices M and N are are similar over the field $\boldsymbol{k}$ if there exists an invertible matrix A with entries in $\boldsymbol{k}$ such that $M = ANA^{-1}$.

[40] see 2.96.

[41] see 2.60.1 on page 36

Observation 4.6. *Let $\boldsymbol{k}$ be a field and M and N be $n \times n$ matrices with entries in $\boldsymbol{k}$. Let $\phi : \boldsymbol{k}^n \to \boldsymbol{k}^n$ be the linear transformation $\phi(v) = Mv$ for all $v \in \boldsymbol{k}^n$. Then, the matrices M and N are similar if and only if there is a basis $\mathscr{B}$ for $\boldsymbol{k}^n$ such that N is the matrix of ϕ with respect to $\mathscr{B}$.*

Proof. If this looks interesting, then write a proof. □

Problem 4.7. I think of "canonical forms" as solving the following problem. I have a linear transformation from a finite dimensional vector space to itself. What basis should I use in order to make the matrix as easy as possible?

I can imagine some one else solving the following problem. I have two $n \times n$ matrices. How can I decide if they are similar?

Data 4.8. Let $\boldsymbol{k}$ be a field, V be a finite dimensional vector space over $\boldsymbol{k}$, and $T : V \to V$ be a linear transformation. View V as a $\boldsymbol{k}[x]$-module by $xv = T(v)$ for all $v \in V$. What does Theorem 4.3 tell us?

Example 4.9. Adopt the data of 4.8. If V is a cyclic $\boldsymbol{k}[x]$-module, generated by v_0 and

$$V \cong \frac{\boldsymbol{k}[x]}{(\alpha_0 + \alpha_1 x + \dots \alpha_4 x^4 + x^5)},$$

then one basis for V is $v_0, T(v_0), T^2(v_0), T^3(v_0), T^4(v_0)$. It follows that the matrix for T is

$$\begin{array}{c|ccccc}
 & v_0 & T(v_0) & T^2(v_0) & T^3(v_0) & T^4(v_0) \\
\hline
v_0 & 0 & 0 & 0 & 0 & -\alpha_0 \\
T(v_0) & 1 & 0 & 0 & 0 & -\alpha_1 \\
T^2(v_0) & 0 & 1 & 0 & 0 & -\alpha_2 \\
T^3(v_0) & 0 & 0 & 1 & 0 & -\alpha_3 \\
T^4(v_0) & 0 & 0 & 0 & 1 & -\alpha_4
\end{array}$$

The matrix is called the companion matrix for the monic polynomial $\alpha_0 + \alpha_1 x + \dots \alpha_4 x^4 + x^5$. If the $\boldsymbol{k}[x]$-module V is isomorphic to

$$\frac{\boldsymbol{k}[x]}{(f_1)} \oplus \cdots \oplus \frac{\boldsymbol{k}[x]}{(f_s)},$$

with $f_1 | \cdots | f_s$ in $\boldsymbol{k}[x]$, then the Rational Canonical Form for T, where $T(v) = xv$ for all $v \in V$, is

$$\begin{bmatrix} C(f_1) & 0 & 0 & \cdots & 0 \\ 0 & C(f_2) & 0 & \cdots & 0 \\ 0 & 0 & C(f_3) & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & C(f_s) \end{bmatrix}, \tag{4.9.1}$$

where $C(f_i)$ is the companion matrix for f_i. The Rational Canonical Form of a matrix is completely unique and it does not change if one moves to a bigger field.

Example 4.10. Adopt the data of 4.8. If V is a cyclic $\boldsymbol{k}[x]$-module, generated by v_0 and

$$V \cong \frac{\boldsymbol{k}[x]}{(x^5)},$$

then one basis for V is $v_0, T(v_0), T^2(v_0), T^3(v_0), T^4(v_0)$. It follows that the matrix for T is

$$\begin{array}{c|ccccc} & v_0 & T(v_0) & T^2(v_0) & T^3(v_0) & T^4(v_0) \\ \hline v_0 & 0 & 0 & 0 & 0 & 0 \\ T(v_0) & 1 & 0 & 0 & 0 & 0 \\ T^2(v_0) & 0 & 1 & 0 & 0 & 0 \\ T^3(v_0) & 0 & 0 & 1 & 0 & 0 \\ T^4(v_0) & 0 & 0 & 0 & 1 & 0 \end{array}$$

If V is a cyclic $\boldsymbol{k}[x]$-module, generated by v_0 and

$$V \cong \frac{\boldsymbol{k}[x]}{(x-a)^5},$$

then one basis for V is $v_0, (T-a)(v_0), (T-a)^2(v_0), (T-a)^3(v_0), (T-a)^4(v_0)$. It follows that the matrix for T is

$$\begin{array}{c|ccccc} & v_0 & (T-a)(v_0) & (T-a)^2(v_0) & (T-a)^3(v_0) & (T-a)^4(v_0) \\ \hline v_0 & a & 0 & 0 & 0 & 0 \\ (T-a)(v_0) & 1 & a & 0 & 0 & 0 \\ (T-a)^2(v_0) & 0 & 1 & a & 0 & 0 \\ (T-a)^3(v_0) & 0 & 0 & 1 & a & 0 \\ (T-a)^4(v_0) & 0 & 0 & 0 & 1 & a \end{array}$$

The matrix is called the 5×5 Jordan Block associated to a. This matrix is denoted $J_5(a)$. If the $\boldsymbol{k}[x]$-module V is isomorphic to

$$\frac{\boldsymbol{k}[x]}{(f_1)} \oplus \cdots \oplus \frac{\boldsymbol{k}[x]}{(f_s)},$$

with $f_i = (x-a_i)^{n_i}$, then the Jordan Canonical Form for T, where $T(v) = xv$ for all $v \in V$, is

$$\begin{bmatrix} J_{n_1}(a_1) & 0 & 0 & \cdots & 0 \\ 0 & J_{n_2}(a_2) & 0 & \cdots & 0 \\ 0 & 0 & J_{n_3}(a_3) & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & J_{n_s}(a_s) \end{bmatrix}. \tag{4.10.1}$$

The Jordan Canonical Form of a matrix is unique up to the order of the Jordan blocks. **I have only described the Jordan canonical form of a matrix in the case that the field is algebraically closed.**

Remarks 4.11. Let $\boldsymbol{k}$ be a field, V be a finite dimensional vector space over $\boldsymbol{k}$ and $T : V \to V$ be a linear transformation.

(a) The characteristic polynomial of T is $\det(x\,\mathrm{id}_V - T)$. One can make a matrix M for $x\,\mathrm{id}_V - T$ with respect to some basis $v_1, \ldots, v_n$:

$$M = \begin{bmatrix} m_{11} & \ldots & m_{1n} \\ \vdots & \vdots & \vdots \\ m_{n1} & \ldots & m_{nn} \end{bmatrix}$$

with $(x\,\mathrm{id}_V - T)(v_j) = \sum_{i=1}^n m_{ij} v_i$. (Each m_{ij} is an element of $\boldsymbol{k}[x]$.) The characteristic polynomial of T is the determinant of M. One can use any basis because $\det AB = (\det A)(\det B)$ for

$n \times n$ matrices A and B with entries in a (commutative) ring. (I do not intend to prove this.) If M and M' both are matrices for $x\,\mathrm{id}_V - T$, then there is an invertible matrix A with entries in $\boldsymbol{k}$ such that $M = AM'A^{-1}$. It follows that $\det M = \det M'$.

(b) The element a of $\boldsymbol{k}$ is an eigenvalue of T if and only if a is a root of the characteristic polynomial of T.

(c) If f is a monic polynomial in $\boldsymbol{k}[x]$, then the characteristic polynomial of the companion matrix $C(f)$ is f.

Proof. Use induction on the degree of f. If $\deg f = 1$, then $f = x + c_0$, $C(f) = \begin{bmatrix}-c_0\end{bmatrix}$, the characteristic polynomial of f is $\det\begin{bmatrix}x + c_0\end{bmatrix} = x + c_0$. If $\deg f = 2$, then $f = x^2 + c_1x + c_0$, $C(f) = \begin{bmatrix}0 & -c_0\\ 1 & -c_1\end{bmatrix}$, the characteristic polynomial of f is

$$\det\begin{bmatrix}x & +c_0\\ -1 & x + c_1\end{bmatrix} = x(x + c_1) + c_0.$$

Similarly, to compute the characteristic polynomial of $C(x^3 + c_2x^2 + c_1x + c_0)$, we expand across the first row and obtain

$$\det\begin{bmatrix}x & 0 & c_0\\ -1 & x & c_1\\ 0 & -1 & x + c_2\end{bmatrix} = x \det \underbrace{\begin{bmatrix}x & c_1\\ -1 & x + c_2\end{bmatrix}}_{\text{the companion matrix of } \frac{f - c_0}{x}} \underbrace{-}_{(-1)^{n-1}} c_0 \det \underbrace{\begin{bmatrix}-1 & x\\ 0 & -1\end{bmatrix}}_{(-1)^{n-1}}$$

$$= x\frac{f - c_0}{x} + c_0 = f.$$

□

(d) The characteristic polynomial of the Jordan block $J_e(a)$ is $(x - a)^e$. Indeed,

$$J_3(a) = \begin{bmatrix}a & 0 & 0\\ 1 & a & 0\\ 0 & 1 & a\end{bmatrix}$$

and the characteristic polynomial of $J_3(a)$ is

$$\det\begin{bmatrix}x - a & 0 & 0\\ -1 & x - a & 0\\ 0 & -1 & x - a\end{bmatrix} = (x - a)^3.$$

(e) The characteristic polynomial of the matrix (4.9.1) is $\prod_{i=1}^{s} f_i$. The characteristic polynomial of the matrix (4.10.1) is $\prod_{i=1}^{s}(x - a_i)^{e_i}$. The minimal polynomial of (4.9.1) is f_s. If $c(x)$ is the characteristic polynomial of the linear transformation T, then $c(T) = 0$. (This fact is called the Cayley-Hamilton Theorem.)

5. Galois Theory

5.A. **Preliminary material.**

Example 5.1. Let $f = x^3 - 4x^2 + 6x - 2$ in $\mathbb{Q}[x]$.

(a) Show that f is irreducible in $\mathbb{Q}[x]$.
(b) Show that $K = \mathbb{Q}[x]/(f)$ is a field.
(c) What is $\dim_{\mathbb{Q}} K$?
(d) What is a good basis for K over $\mathbb{Q}$?
(e) Write $\frac{1}{\bar{x}}$ in terms of the basis of (d).?
(f) Write $\frac{1}{\bar{x}+1}$ in terms of the basis of (d).?

(a) I propose that we use the Eisenstein criterion. (It is a Homework problem.)

Let $f = a_0 + \cdots + a_n x^n$ be a primitive polynomial in $\mathbb{Z}[x]$. Suppose there is a prime integer p with $p|a_i$ for $0 \le i \le n-1$, but p^2 does not divide a_0 and p does not divide a_n. Then f is an irreducible polynomial in $\mathbb{Q}[x]$.

We see that 2 divides every coefficient except the leading coefficient and 4 does not divide the constant term. We conclude that f is irreducible.

(a) In a PID like $\mathbb{Q}[x]$ the following three statements are equivalent about an element f:

(i) The element f is irreducible.
(ii) The ideal (f) is maximal.
(iii) The ideal (f) is prime.

(c) and (d) The dimension of K as a $\mathbb{Q}$ vector space is 3 and $\bar{1}$, $\bar{x}$, and $\overline{x^2}$ is a basis.

(e). We know that $\bar{x}^3 - 4\bar{x}^2 + 6\bar{x} - 2 = 0$. Thus, $(1/2)(\bar{x}^2 - 4\bar{x} + 6) = 1/\bar{x}$.

(f). Use the Euclidean Algorithm to write $A(x^3 - 4x^2 + 6x - 2) + B(x+1) = 1$ in $\mathbb{Q}[x]$. Then $B = \frac{1}{x+1}$ in K.

Observation 5.2. *Let $\boldsymbol{k} \subseteq K$ be a field extension, let $u \in K$, and let $\phi : \boldsymbol{k}[x] \to \boldsymbol{k}[u]$ be the ring homomorphism $\phi(f(x)) = f(u)$.*

(a) *If u is transcendental over k, then ϕ is a ring isomorphism and ϕ induces a field isomorphism $\boldsymbol{k}(x) \to \boldsymbol{k}(u)$.*
(b) *If u is algebraic over $\boldsymbol{k}$ and $f(x)$ is the minimal polynomial of u over $\boldsymbol{k}$, then ϕ induces an isomorphism*

$$\frac{\boldsymbol{k}[x]}{(f)} \to \boldsymbol{k}[u]$$

and $\boldsymbol{k}[u] = \boldsymbol{k}(u)$.

I think we did this. At any rate, it is easy.

Observation 5.3. *If $\boldsymbol{k} \subseteq K \subseteq E$ are finite dimensional fields extensions, then*

$$\dim_{\boldsymbol{k}} E = \dim_{\boldsymbol{k}} K \dim_K E.$$

Proof. Let $u_1, \dots, u_n$ be a basis for K over $\boldsymbol{k}$ and $v_1, \dots, v_m$ be a basis for E over K. It suffices to show that

$$\{u_i v_j | 1 \le i \le n \text{ and } 1 \le j \le m\}$$

is a basis for E over $\boldsymbol{k}$.

Span. If $e \in E$, then $e = \sum_{j=1}^m \alpha_j v_j$ for some $\alpha_j \in K$. Furthermore, each α_j is equal to $\sum_{i=1}^n \beta_{ij} u_i$ for some $\beta_{ij} \in \boldsymbol{k}$ and

$$e = \sum_{j=1}^m \alpha_j v_j = \sum_{j=1}^m \sum_{i=1}^n \beta_{ij} u_i v_j,$$

as desired.

Linearly Independent. Suppose β_{ij} are elements of $\boldsymbol{k}$, for $1 \le i \le n$ and $1 \le j \le m$ and

$$\sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} \beta_{ij} u_i v_j = 0.$$

Thus,

$$\sum_{j=1}^m \underbrace{\left(\sum_{i=1}^n \beta_{ij} u_i \right)}_{\in K} v_j = 0.$$

The v's are linearly independent over K; so, each $\sum_{i=1}^n \beta_{ij} u_i = 0$. The u's are linearly independent over $\boldsymbol{k}$ and the β_{ij} are in $\boldsymbol{k}$. We conclude that each $\beta_{ij} = 0$. □

5.B. **Ruler and Compass Construction.**

<u>The rules</u> Pick two points on the plane. Call one "0" and call the other one "1". You can draw the line between two constructed points. You can draw a circle with a previously constructed center and radius. The intersection of two drawings is constructible. Let

$$\mathscr{C} = \{z \in \mathbb{C} | z \text{ is constructible}\}.$$

Example 5.4. Here are some constructible objects. First of all, each element of $\mathbb{Z}$ is constructible. The pictures on the next page justify the following assertions.

(a) If the line ℓ is constructible, and P is a constructible point on ℓ, then the line through P perpendicular to ℓ is constructible.

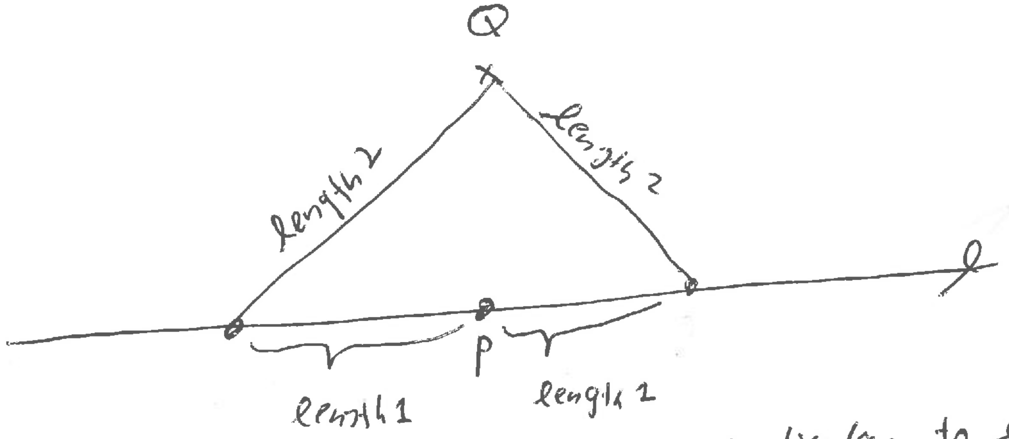
(b) If the line ℓ is constructible, and P is a constructible point not on ℓ, then the line through P perpendicular to ℓ is constructible.

(c) If the line ℓ is constructible, and P is a constructible point not on ℓ, then the line through P parallel to ℓ is constructible. (In particular, every Gaussian integer is constructible.)

(d) If the angle θ is constructible, then the angle $\theta/2$ is constructible.

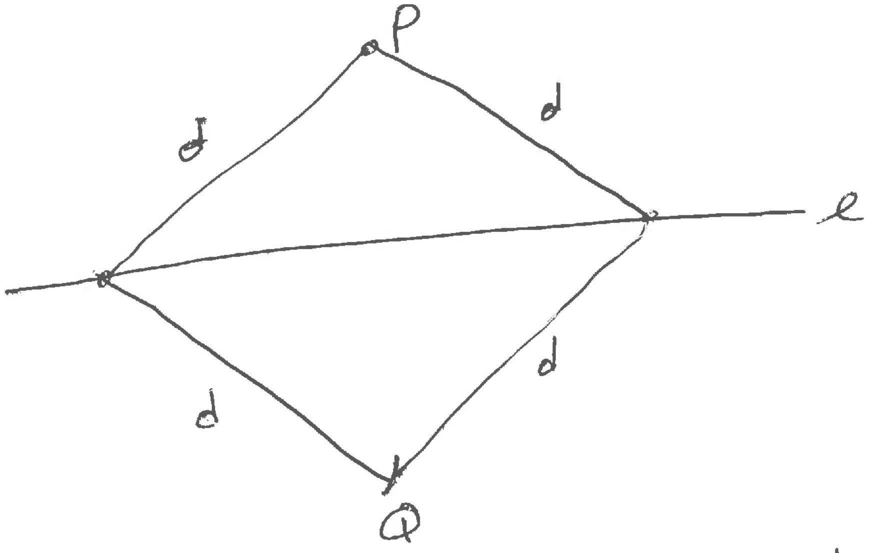
Some Constructible Objects

(a)



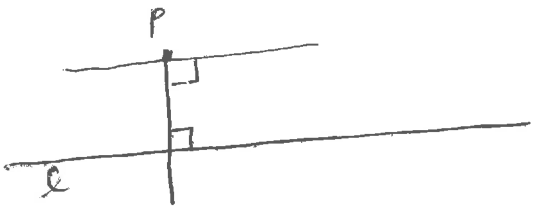
The line containing PQ is perpendicular to ℓ

(b)

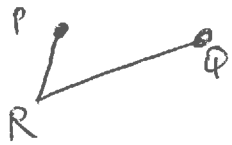


The line containing P and Q is perpendicular to ℓ

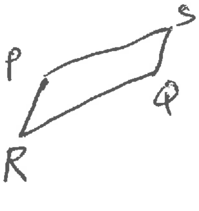
(c) Use (b) Then (a)



(d)



use b twice



The angle PRS bisects the angle PRQ

Goal 5.5. There are four classical problems about ruler and compass constructions.

(a) Find a ruler and compass algorithm for trisecting the angle.
(b) Find a ruler and compass algorithm for squaring the circle.[42]
(c) Find a ruler and compass algorithm for doubling the cube.
(d) Which regular n-gons can be constructed using ruler and compass?

Here is the main result we use to attack Goal 5.5

Theorem 5.6. *If $u \in \mathscr{C}$, then $\dim_{\mathbb{Q}} \mathbb{Q}[u] = 2^n$ for some n.*

We will take a few steps to prove Theorem 5.6.

Lemma 5.7. *The set of constructible numbers $\mathscr{C}$ is a field which is closed under square root and complex conjugation.*

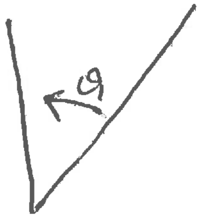
Proof. The set $\mathscr{C}$ is closed under addition. The argument we used for bisecting an angle may be used here.

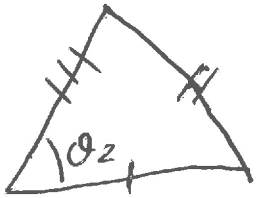
The set $\mathscr{C}$ is closed under multiplication. Write each complex number in the form $re^{i\theta}$, where r and θ are non-negative real numbers. One must be able to add angles and multiply positive real numbers.

[42] We will not actually do anything with this one. It is impossible to find a square with the same area as a circle because π is a transcendental number. One learns that result elsewhere.

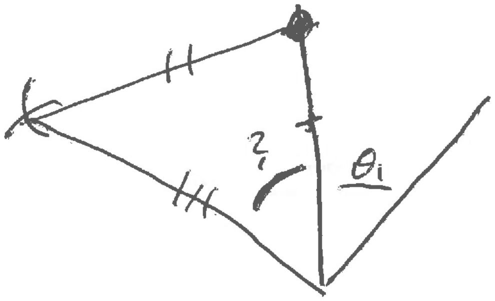
One adds angles by moving a triangle:

If





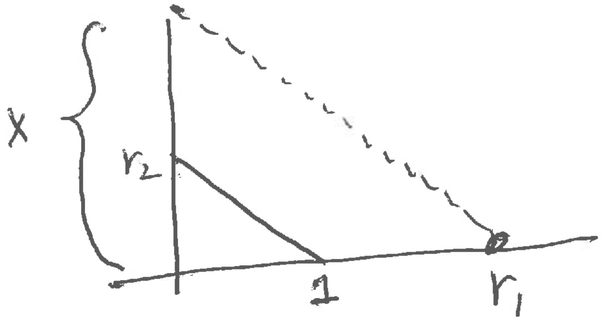
can be constructed then



Conclude from side-side-side that $? = \theta_2$

To multiply positive real numbers

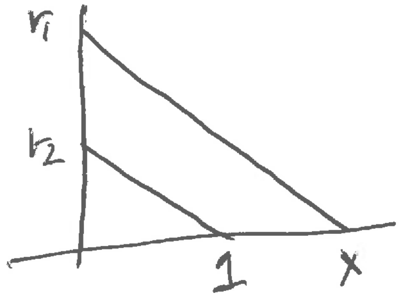
Draw a line parallel to a given line through a given Point



use similar triangles $\frac{X}{r_2} = \frac{r_1}{1}$ ∴ $X = r_1 r_2$

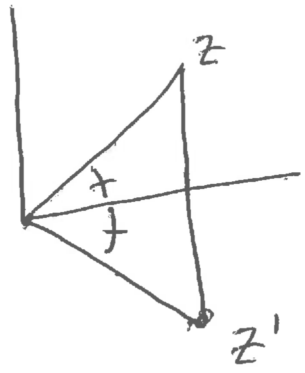
Divide similarly

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$



$$\frac{x}{1} = \frac{r_1}{r_2}$$

Closed under complex conjugation

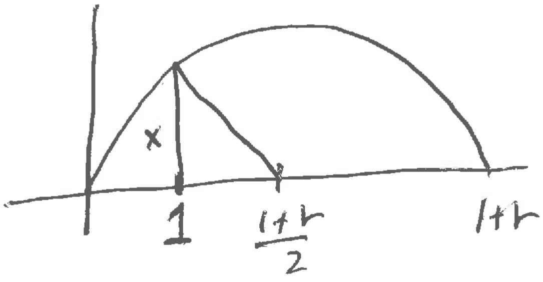


Closed under square root

The square roots of $re^{i\theta}$ are $\pm\sqrt{r}\, e^{i\frac{\theta}{2}}$

We already saw how to bisect an angle.

To Find $\sqrt{r}$



With respect to $\sqrt{r}$ for some positive real number r, we constructed a right triangle whose non-hypotenuse sides are x and $\frac{|r-1|}{2}$. The hypotenuse is $\frac{1+r}{2}$. It follows that

$$x^2 + \left(\frac{|r-1|}{2}\right)^2 = \left(\frac{1+r}{2}\right)^2.$$

Of course, $x^2 = r$. □

I think of the next Lemma as the subtle part of the argument.

Lemma 5.8. *If $\mathscr{C}'$ is a subfield which is closed under conjugation and taking square root, then $\mathscr{C} \subseteq \mathscr{C}'$.*

Proof. The subtle part of the argument occurs right away. It suffices to prove that

$$\begin{aligned} &L_1 \cap L_2 \in \mathscr{C}',\\ &L_1 \cap C_1 \in \mathscr{C}', \text{ and}\\ &C_1 \cap C_2 \in \mathscr{C}', \end{aligned}$$

where L_1 and L_2 are lines through distinct points of $\mathscr{C}'$ and C_1 and C_2 are circles with centers and radii from $\mathscr{C}'$.

Suppose $z \in x + y\mathbb{i} \in \mathscr{C}'$, then $\bar{z} = x - y\mathbb{i} \in \mathscr{C}'$ and x and y are in $\mathscr{C}'$.

If $z_1 = x_1 + y_1\mathbb{i}$ and $z_2 = x_2 + y_2\mathbb{i}$ are in $\mathscr{C}'$ and L is the line through z_1 and z_2, then

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

and $ax + by + c = 0$, with a, b, and c in $\mathscr{C}'$.

Now consider the circle with center $x_0 + y_0\mathbb{i}$ in $\mathscr{C}'$ and radius r in $\mathscr{C}'$. The equation of this circle is

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

$$x^2 - 2x_0x - x_0^2 + y^2 - 2y_0y + y_0^2 = r^2$$

$$x^2 + y^2 + ax + by + c = 0,$$

with a, b, and c in $\mathscr{C}'$.

To find the intersection of two lines, we solve

$$a_1x + b_1y + c_1 = 0 \quad \text{and} \quad a_2x + b_2y + c_2 = 0$$

simultaneously, with $a_1, a_2, b_1, b_2, c_1, c_2$ all in $\mathscr{C}'$. Both coordinates of the solution are in the field $\mathscr{C}'$.

To find the intersection of the line $y = mx + b$ and the circle $x^2 + y^2 + Ax + By + C = 0$, with m, b, A, B, C in $\mathscr{C}'$, we solve

$$x^2 + (mx + b)^2 + Ax + B(mx + b) + C = 0.$$

We solve

$$\alpha x^2 + \beta x + \gamma = 0,$$

with α, β, and γ all in $\mathscr{C}'$. If there is a real solution, then this solution is

$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

and the solution is in $\mathscr{C}'$.

To find the intersection of the circles

$$x^2 + y^2 + A_1x + B_1y + C_1 = 0 \quad \text{and} \quad x^2 + y^2 + A_2x + B_2y + C_2 = 0,$$

with A_i, B_i, and C_i in $\mathscr{C}'$, it suffices to find the intersection of the circle $x^2+y^2+A_1x+B_1y+C_1 = 0$ together with the line $(A_2 - A_1)x + (B_2 - B_1)y + (C_2 - C_1) = 0$. We already saw that this solution is in $\mathscr{C}'$. □

Lemma 5.9. *Let*

$$\mathscr{C}'' = \{u \in \mathbb{C} | \exists u_1, \dots, u_n \in \mathbb{C} \textit{ with } u \in \mathbb{Q}[u_1, \dots u_n] \textit{ and } u_i^2 \in \mathbb{Q}[u_1, \dots u_{i-1}]\}.$$

Then $\mathscr{C}''$ is a subfield of $\mathbb{C}$ closed under square root and conjugation.

Proof.

- If $u \in \mathscr{C}''$, then $\sqrt{u} \in \mathscr{C}''$.

(Indeed, $u \in \mathbb{Q}[u_1, \dots, u_n]$ with $u_i^2 \in \mathbb{Q}[u_1, \dots u_{i-1}]$; so $\sqrt{u} \in \mathbb{Q}[u_1, \dots, u_n, \sqrt{u}]$.)

- If $u \in \mathscr{C}''$, then $u \in \mathbb{Q}[u_1, \dots, u_n]$ with $u_i^2 \in \mathbb{Q}[u_1, \dots u_{i-1}]$ and $\bar{u} \in \mathbb{Q}[\bar{u}_1, \dots, \bar{u}_n]$ with $\bar{u}_i^2 \in \mathbb{Q}[\bar{u}_1, \dots \bar{u}_{i-1}]$.
- If $u, u' \in \mathscr{C}''$, then $u + u'$, uu', and (if $u' \neq 0$) $\frac{u}{u'}$ are all $\mathscr{C}''$.

Indeed, there are $u_1, \dots, u_n$ and $u'_1, \dots, u'_{n'}$ with $u_i^2 \in \mathbb{Q}[u_1, \dots, u_{i-1}]$, $u_i'^2 \in \mathbb{Q}[u'_1, \dots, u'_{i-1}]$, $u \in \mathbb{Q}[u_1, \dots, u_n]$, and $u' \in \mathbb{Q}[u'_1, \dots, u'_{n'}]$. Observe that

$$\mathbb{Q}[u_1, \dots, u_n, u'_1, \dots, u'_{n'}]$$

is a subfield of $\mathscr{C}''$ which contains both u and u'. □

Lemma 5.10. *The field $\mathscr{C}''$ from Lemma* 5.9 *is equal to the field $\mathscr{C}$ of constructible numbers.*

Proof. Combine Lemmas 5.8 and 5.9 to see that $\mathscr{C} \subseteq \mathscr{C}''$.

We show $\mathscr{C}'' \subseteq \mathscr{C}$. Take $u \in \mathscr{C}''$. It follows that there exist $u_1, \dots, u_n \in \mathbb{C}$ with $u \in \mathbb{Q}[u_1, \dots u_n]$ and $u_i^2 \in \mathbb{Q}[u_1, \dots u_{i-1}]$. We know from Lemma 5.7 that $\mathscr{C}$ is a field which is closed under the taking of square root. It follows that $u_1 \in \mathscr{C}$. It also follows that $u_2 \in \mathscr{C}$. etc. Indeed, all of $\mathbb{Q}[u_1, \dots u_n] \subseteq \mathscr{C}$. In particular, $u \in \mathscr{C}$. □

Theorem. 5.6 *If $u \in \mathscr{C}$, then* $\dim_{\mathbb{Q}} \mathbb{Q}[u] = 2^n$ *for some n.*

Proof. We know from Lemma 5.10 that $u \in \mathbb{Q}[u_1, \dots, u_r]$ and $\dim_{\mathbb{Q}} \mathbb{Q}[u_1, \dots, u_r] = 2^s$ for some $s \le r$. (Here is the argument: If $u_i^2 \in \mathbb{Q}[u_1, \dots, u_{i-1}]$, then the minimal polynomial of u_i over $\mathbb{Q}[u_1, \dots, u_{i-1}]$ has degree 1 or 2. Hence the dimension of $\mathbb{Q}[u_1, \dots, u_i]$ as a vector space over $\mathbb{Q}[u_1, \dots, u_{i-1}]$ is either one or two.) Apply Lemma 5.3

$$\dim_{\mathbb{Q}} \mathbb{Q}[u] \dim_{\mathbb{Q}[u]} \mathbb{Q}[u_1, \dots, u_r] = \dim_{\mathbb{Q}} \mathbb{Q}[u_1, \dots, u_r] = 2^s.$$

All of the factors of 2^s have the form 2^n for some n. □

Corollary 5.11. *There does not exist a ruler and compass algorithm for trisecting an angle.*

Proof. The angle 60 degrees is constructible. (It is the angle in an isosceles or it lives in a right triangle with hypotenuse 1 and one of the sides $1/2$.) If there were a ruler and compass algorithm for trisecting an angle, then $\cos 20°$ would be constructible. We will prove that the minimal polynomial for $\cos 20°$ has degree three. This will prove that $\cos 20°$ is not constructible (by Theorem 5.6).

It would be nice to know $\cos 3\theta$ in terms of powers of $\cos\theta$. It turns out that we do know this. Indeed,

$$\cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) + i(\text{stuff}) = \Big(\cos(\theta) + i\sin(\theta)\Big)^3 = (e^{i\theta})^3 = e^{i3\theta} = \cos(3\theta) + i\sin(3\theta).$$

Thus,

$$\begin{aligned}\cos(3\theta) &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

Let $a = \cos 20°$. We have shown that a satisfies

$$1/2 = 4x^3 - 3x.$$

In other words a satisfies

$$f(x) = 8x^3 - 6x - 1.$$

It turns out that f is irreducible in $\mathbb{Q}[x]$. It suffices to show that $f(\frac{x}{2}) = \underbrace{x^3 - 3x - 1}_{g(x)}$ is irreducible in $\mathbb{Q}[x]$. The polynomial $g(x)$ is cubic. It follows that if $g(x)$ can be factored over $\mathbb{Q}$, then g has a rational root. The only possible rational roots of a polynomial with integer coefficients are

$$\frac{\text{integer factors of the constant term}}{\text{integer factors of the leading coefficient}}.$$

The only possible rational roots of g are ± 1 and neither one is a root. We conclude that

$$\dim_{\mathbb{Q}} \mathbb{Q}[\cos 20°] = 3$$

and therefore, $\cos 20°$ is not a constructible number. □

Corollary 5.12. *It is impossible to "duplicate the cube" using ruler and compass.* (*In other words, there is no ruler and compass algorithm for constructing a cube with twice the volume of a given cube.*)

Proof. It is possible to construct a cube with volume one. We will prove that it is not possible to construct a cube of volume 2. A cube of volume 2 would have edges of length equal to $\sqrt[3]{2}$. The minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$ is x^3-2. (Use Eisenstein's criteria to see that x^3-2 is irreducible over $\mathbb{Q}$.) □

Corollary 5.13. *If p is a prime integer and a regular p-gon is constructible using ruler and compass, then $p = 2^{2^t} + 1$ for some t.*

Remarks. (a) A prime integer of the form $p = 2^{2^t} + 1$ is called a Fermat prime.

(b) The complete answer is: A regular n-gon is constructible using ruler and compass if and only if $n = 2^e p_2 \cdots p_s$, with $0 \leq e$, and the p_i are distinct Fermat primes. We will prove this result later in the course.[43]

(c) The only known Fermat primes are $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, and $F_4 = 65537$.

Proof. If a regular n-gon is constructible, then $\omega = e^{(2\pi i)/p} \in \mathscr{C}$. The minimal polynomial of ω is

$$\frac{x^p - 1}{x - 1},$$

which has degree $p-1$. Apply Theorem 5.6 to see that $p-1 = 2^s$. We make one further observation. If u is a positive odd integer (with $3 \leq u$), then

$$x^u + 1 = (x + 1)(x^{u-1} - \cdots - x + 1);$$

consequently, if s has any odd factors, say $s = uv$ with $3 \leq u$ odd, then

$$p = 2^s + 1 = (2^v + 1)((2^v)^{u-1} - \cdots - 2^v + 1).$$

Of course, prime integers do not factor and s has only even factors. In other words, $s = 2^t$ for some t and $p = 2^{2^t} + 1$. □

5.C. **The Fundamental Theorem of Galois Theory.**

Definition 5.14.

(a) If $\boldsymbol{k} \subseteq K$ are fields, then

$$\operatorname{Aut}_{\boldsymbol{k}} K = \{\sigma : K \to K \mid \sigma \text{ is a field automorphism and } \sigma|_{\boldsymbol{k}} \text{ is the identity map}\}.$$

(b) If G is a group of automorphisms of the field K, then

$$K^G = \{u \in K \mid \sigma(u) = u \text{ for all } \sigma \in G\}.$$

(c) The field extension $\boldsymbol{k} \subseteq K$ is called Galois if

$$\boldsymbol{k} = K^G \quad \text{for } G = \operatorname{Aut}_{\boldsymbol{k}} K.$$

Remark 5.15. If $\boldsymbol{k} \subseteq K$ are fields[44] and $\boldsymbol{k} = K^G$ for some group of automorphisms G of K, then $\boldsymbol{k} \subseteq K$ is a Galois extension of fields.

Proof. Observe that $G \subseteq \operatorname{Aut}_{\boldsymbol{k}} K$. It follows that

$$\boldsymbol{k} \subseteq K^{\operatorname{Aut}_{\boldsymbol{k}} K} \subseteq K^G = \boldsymbol{k}.$$

□

Remark 5.16. If $\boldsymbol{k} \subseteq K$ are fields, then

$$\boldsymbol{k} \subseteq K^{\operatorname{Aut}_{\boldsymbol{k}} K}$$

holds automatically. To show that the field extension is Galois, one need only show that every element of $K \setminus \boldsymbol{k}$ is moved by some element of $\operatorname{Aut}_{\boldsymbol{k}} K$.

[43] See Corollary 5.89.

[44] In this result $\dim_{\boldsymbol{k}} K$ could be infinite.

Questions 5.17.

(a) Compute $\mathrm{Aut}_{\mathbb{Q}}\,\mathbb{Q}[i]$. Is $\mathbb{Q} \subseteq \mathbb{Q}[i]$ Galois?
(b) Compute $\mathrm{Aut}_{\mathbb{Q}}\,\mathbb{Q}[\sqrt[3]{2}]$. Is $\mathbb{Q} \subseteq \mathbb{Q}[\sqrt[3]{2}]$ Galois?
(c) Compute $\mathrm{Aut}_{\mathbb{Q}}\,\mathbb{Q}[\omega]$, for $\omega = e^{(2\pi i)/7}$. Is $\mathbb{Q} \subseteq \mathbb{Q}[\omega]$ Galois?

Before we attack the questions, here is the key observation.

Observation 5.18. *If $\sigma \in \mathrm{Aut}_{\boldsymbol{k}} K$, $u \in K$, $f \in \boldsymbol{k}[x]$, and $f(u) = 0$, then $f(\sigma(u)) = 0$.*

Proof. Suppose $f = a_n x^n + \cdots + a_0$. Then

$$0 = a_n u^n + \cdots + a_0 \quad \text{in } K.$$

Apply the automorphism σ. Keep in mind that σ acts like the identity map on $\boldsymbol{k}$. Observe that

$$0 = a_n(\sigma(u))^n + \cdots + a_0.$$

Thus $f(\sigma(u))$ is indeed zero. □

Attack Question 5.17.(a) If $\sigma \in \mathrm{Aut}_{\mathbb{Q}}\,\mathbb{Q}[i]$, then σ leaves $\mathbb{Q}$ alone. We need only figure out what σ does to i. We apply Observation 5.18. The number i satisfies the equation $x^2 + 1 = 0$. The only other root of this equation is $-i$. The complex number $-i$ is in $\mathbb{Q}[i]$ So the only possible elements of $\mathrm{Aut}_{\mathbb{Q}}\,\mathbb{Q}[i]$ are the identity map and complex conjugation. Both of these are legitimate maps. We conclude that

$$\mathrm{Aut}_{\mathbb{Q}}\,\mathbb{Q}[i] = \{\mathrm{id}, \bar{\ }\}.$$

We also observe that $\overline{a + bi}$ is different than $a+bi$ unless $b = 0$. We conclude that $\mathbb{Q}[i]^{\mathrm{Aut}_{\mathbb{Q}}\mathbb{Q}[i]} = \mathbb{Q}$. Thus,

$$\mathbb{Q} \subseteq \mathbb{Q}[i]$$

is a Galois extension of fields.

Attack Question 5.17.(b) Again, we must figure out what an element of $\mathrm{Aut}_{\mathbb{Q}}\,\mathbb{Q}[\sqrt[3]{2}]$ does to $\sqrt[3]{2}$. The minimal polynomial of $\sqrt[3]{2}$ is $x^3 - 2$. The other roots of this polynomial are $\omega\sqrt[3]{2}$ and $\omega^2\sqrt[3]{2}$, where $\omega = e^{(2\pi i)/3}$. The only root of $x^3 - 2$ that is also in the field $\mathbb{Q}[\sqrt[3]{2}]$ is $\sqrt[3]{2}$. Thus,

$$\mathrm{Aut}_{\mathbb{Q}}\,\mathbb{Q}[\sqrt[3]{2}] = \{\mathrm{id}\}.$$

Of course,

$$\mathbb{Q}[\sqrt[3]{2}]^{\{\mathrm{id}\}} = \mathbb{Q}[\sqrt[3]{2}].$$

Thus,

$$\mathbb{Q} \subseteq \mathbb{Q}[\sqrt[3]{2}]$$

is NOT a Galois extension.

Attack Question 5.17.(c) The minimal polynomial of $\omega = e^{(2\pi i)/7}$ is $\frac{x^7-1}{x-1} = g(x)$ by the Eisenstein trick. The other roots of $g(x)$ in $\mathbb{C}$ are $\omega^2, \omega^3, \omega^4, \omega^5, \omega^6$. All of these roots are in $K = \mathbb{Q}[\omega]$.

We better prove the second key observation before we finish answering this question.

Observation 5.19. *If $\boldsymbol{k} \subset K_1$ and $\boldsymbol{k} \subseteq K_2$ are field extensions, $u_i \in K_i$, and u_1 and u_2 have the same minimal polynomial over $\boldsymbol{k}$, then the fields $\boldsymbol{k}[u_1]$ and $\boldsymbol{k}[u_2]$ are isomorphic and there is an isomorphism between the two fields which sends $g(u_1) \in K_1$ to $g(u_2) \in K_2$, for all polynomials $g(x) \in \boldsymbol{k}[x]$.*

Proof. Let $f(x) \in \boldsymbol{k}[x]$ be the common minimal polynomial of u_1 and u_2. The first isomorphism theorem yields isomorphisms

$$\phi_1 : \boldsymbol{k}[x]/(f) \to \boldsymbol{k}[u_1] \quad \phi_2 : \boldsymbol{k}[x]/(f) \to \boldsymbol{k}[u_2],$$

with $\phi_i(\bar{x}) = u_i$. The composition $\phi_2 \circ \phi_1^{-1}$ is an isomorphism from $\boldsymbol{k}[u_1]$ to $\boldsymbol{k}[u_2]$, which sends $g(u_1) \in K_1$ to $g(u_2) \in K_2$, for all polynomials $g(x) \in \boldsymbol{k}[x]$. □

Return to Question 5.17.(c) Apply Observation 5.19 to see that for each integer a with $1 \le a \le 6$, there is a $\mathbb{Q}$-algebra isomorphism

$$\sigma_a : \mathbb{Q}[\omega] \to \mathbb{Q}[\omega^a]$$

which is given by $\sigma_a(\omega) = \omega^a$.

We notice further that $\mathbb{Q}[\omega^a] \subseteq \mathbb{Q}[\omega]$ with

$$\dim_{\mathbb{Q}} \mathbb{Q}[\omega^a] = \dim_{\mathbb{Q}} \mathbb{Q}[\omega].$$

It follows that each

$$\sigma_a : \mathbb{Q}[\omega] \to \mathbb{Q}[\omega]$$

is an automorphism.

We have shown that $\operatorname{Aut}_{\mathbb{Q}} \mathbb{Q}[\omega]$ has exactly six elements σ_a for $1 \le a \le 6$. There are only two isomorphism classes of groups of order 6, namely $\mathbb{Z}/(6)$ and S_3. We see that $\operatorname{Aut}_{\mathbb{Q}} \mathbb{Q}[\omega]$ is Abelian

$$(\sigma_a \circ \sigma_b)(\omega) = \sigma_a(\omega^b) = (\omega^a)^b.$$

(Or maybe we observe that the arithmetic among the σ_a acts as though the a's are the units in the multiplicative group $(\frac{\mathbb{Z}}{(7)} \setminus \{0\}, \times)$. We know (from a HW problem in set three, for example,) this group is cyclic.) The automorphism σ_2 does not generate $\operatorname{Aut}_{\mathbb{Q}} \mathbb{Q}[\omega]$ because $\sigma_2^3 = \mathrm{id}$. Observe that σ_3 generates $\operatorname{Aut}_{\mathbb{Q}} \mathbb{Q}[\omega]$:

$$\sigma_3(\omega) = \omega^3, \quad \sigma_3^2(\omega) = \omega^2, \quad \sigma_3^3(\omega) = \omega^6, \quad \sigma_3^4(\omega) = \omega^4, \quad \sigma_3^5(\omega) = \omega^5, \quad \sigma_3^6(\omega) = \omega.$$

Finally we ask if the extension $\mathbb{Q} \subseteq \mathbb{Q}[\omega]$ is Galois. Let

$$u = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + a_5\omega^5.$$

Let $\sigma = \sigma_3$. We ask if σ moves u. We see that

$$\begin{aligned} \sigma(u) &= a_0 + a_1\omega^3 + a_2\omega^6 + a_3\omega^2 + a_4\omega^5 + a_5\omega \\ &= \left\{ \begin{array}{l} a_0 + a_5\omega + a_3\omega^2 + a_1\omega^3 \qquad\qquad + a_4\omega^5 \\ -a_2 - a_2\omega - a_2\omega^2 - a_2\omega^3 - a_2\omega^4 - a_2\omega^5. \end{array} \right. \end{aligned}$$

If $a_2 \neq 0$, then σ moves u, Henceforth, take $a_2 = 0$. Observe that σ moves u unless $a_5 = a_1$ and $a_3 = a_2$ and $a_3 = a_1$ and $a_4 = 0$, and $a_4 = a_5$. In other words, σ moves u unless $u \in \mathbb{Q}$. We conclude that $\mathbb{Q}[\omega]^{\langle\sigma\rangle} = \mathbb{Q}$. Thus, the field extension $\mathbb{Q} \subseteq \mathbb{Q}[\omega]$ is a Galois extension.

Theorem 5.20. The Fundamental Theorem of Galois Theory. *Let $\boldsymbol{k} \subseteq K$ be a finite dimensional Galois extension and let $G = \operatorname{Aut}_{\boldsymbol{k}} K$. The following statements hold.*

(a) *There is a one-to-one (inclusion reversing) correspondence between the subgroups of G and the fields E with $\boldsymbol{k} \subseteq E \subseteq K$. The correspondence sends*

$$H \longrightarrow K^H$$
$$\operatorname{Aut}_E K \longleftarrow E$$

(b) *If H is a subgroup of G, then $|H| = \dim_{K^H} K$.*

(c) *The field K is Galois over each intermediate field E, with $\boldsymbol{k} \subseteq E \subseteq K$.*

(d) *If E is an intermediate field, then*

$$E \text{ is Galois over } \boldsymbol{k} \iff \operatorname{Aut}_E K \lhd \operatorname{Aut}_{\boldsymbol{k}} K.$$

Furthermore, in this case,

$$\operatorname{Aut}_{\boldsymbol{k}} E \cong \frac{\operatorname{Aut}_{\boldsymbol{k}} K}{\operatorname{Aut}_E K}.$$

Remark 5.21. To prove (d) we show that if $\boldsymbol{k} \subseteq E \subseteq K$ are finite dimensional field extensions with $\boldsymbol{k} \subseteq E$ and $\boldsymbol{k} \subseteq K$ both Galois extensions, then every element σ of $\operatorname{Aut}_{\boldsymbol{k}} K$ restricts to become an element of $\operatorname{Aut}_{\boldsymbol{k}} E$ and then we apply the first isomorphism theorem to the restriction map $\operatorname{Aut}_{\boldsymbol{k}} K \to \operatorname{Aut}_{\boldsymbol{k}} E$.

Example 5.22. Apply The Fundamental Theorem of Galois Theory to the Galois extension

$$\mathbb{Q} \subseteq \mathbb{Q}[\omega].$$

Let $K = \mathbb{Q}[\omega]$. We know that $G = \operatorname{Aut}_{\mathbb{Q}} K$ is cyclic of order 6 and is generated by σ, where $\sigma(\omega) = \omega^3$. The group G has a total of four subgroups; consequently, the lattice of fields between $\mathbb{Q}$ and K has a total of four fields:

$$\begin{tikzcd}
 & \langle 1\rangle \arrow[dl, hook, "\text{index } 2"'] \arrow[dr, hook, "\text{index } 3"] & \\
\langle\sigma^3\rangle \arrow[dr, hook, "\text{index } 3"'] & & \langle\sigma^2\rangle \arrow[dl, hook, "\text{index } 2"] \\
 & \langle\sigma\rangle &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & K^{\langle 1\rangle} = \mathbb{Q}[\omega] & \\
K^{\langle\sigma^3\rangle} = \mathbb{Q}[?] \arrow[ur, hook, "2", "\text{Gal}"'] & & K^{\langle\sigma^2\rangle} = \mathbb{Q}[?] \arrow[ul, hook, "3"', "\text{Gal}"] \\
 & K^{\langle\sigma\rangle} = \mathbb{Q}. \arrow[ul, hook, "3", "\text{Gal}"'] \arrow[ur, hook, "2"', "\text{Gal}"] &
\end{tikzcd}$$

We want to identify generators for $K^{\langle\sigma^3\rangle}$ and $K^{\langle\sigma^2\rangle}$. We also want to verify that the elements of $\langle\sigma\rangle$ restrict to give automorphisms of $K^{\langle\sigma^3\rangle}$ and $K^{\langle\sigma^2\rangle}$.

Observe that σ^3 carries

$$\begin{aligned}
\omega &\mapsto \sigma^2(\omega^3) = \sigma(\omega^2) = \omega^6\\
\omega^2 &\mapsto \omega^5\\
\omega^3 &\mapsto \omega^4\\
\omega^4 &\mapsto \omega^3\\
\omega^5 &\mapsto \omega^2\\
\omega^6 &\mapsto \omega
\end{aligned}$$

So, σ^3 fixes $\omega+\omega^6$, $\omega^2+\omega^5$, and $\omega^3+\omega^4$. Let $\beta=\omega+\omega^6$. Observe that $\beta^2=\omega^2+2+\omega^5$ and

$$\beta^3 = \omega^3 + 3\omega + 3\omega^6 + \omega^4.$$

Observe that

$$\beta^3+\beta^2-2\beta-1=\sum_{i=0}^{6}\omega^i=0.$$

We have shown that β satisfies the polynomial equation $f(x)=0$, where $f(x)=x^3+x^2-2x-1$. Observe that $f(x)$ is irreducible in $\mathbb{Q}[x]$. Indeed, f is cubic. If f factors in $\mathbb{Q}[x]$, then f has a rational root. The only possible rational roots of a polynomial with integer coefficients are factors of the constant term divided by factors of the leading coefficient. The only possible rational roots of f are ±1. Neither of these is a root. We conclude that f is the minimal polynomial of β over $\mathbb{Q}$. Thus,

$$\mathbb{Q}\subseteq\mathbb{Q}[\beta]\subseteq K^{\langle\sigma^3\rangle}$$

with $\dim_{\mathbb{Q}}\mathbb{Q}[\beta]=3$, because the minimal polynomial of β over $\mathbb{Q}$ has degree 3 and $\dim_{\mathbb{Q}}K^{\langle\sigma^3\rangle}=3$ by Galois theory. Hence

$$\mathbb{Q}[\beta]=K^{\langle\sigma^3\rangle}.$$

It is worth noticing that every element of $\langle\sigma\rangle$ restricts to give an automorphism $\mathbb{Q}[\beta]$. It suffices to show that σ permutes the roots of $f(x)$. Indeed, $\sigma(\beta)=\omega^3+\omega^4$, $\sigma^2(\beta)=\omega^2+\omega^5$, and $\sigma^3(\beta)=\beta$. Furthermore,

$$(x-\beta)(x-\sigma(\beta))(x-\sigma^2(\beta))$$

$$=\begin{cases} x^3\\ -\Big((\omega+\omega^6)+(\omega^3+\omega^4)+(\omega^2+\omega^5)\Big)x^2\\ +\Big((\omega+\omega^6)(\omega^3+\omega^4)+(\omega+\omega^6)(\omega^2+\omega^5)+(\omega^3+\omega^4)(\omega^2+\omega^5)\Big)x\\ -(\omega+\omega^6)(\omega^3+\omega^4)(\omega^2+\omega^5)\end{cases}$$

$$=x^3+x^2-2x-1=f(x).$$

Let's find the generators of $K^{\langle\sigma^2\rangle}$ and let's verify that each element of $\langle\sigma\rangle$ restricts to give an automorphism of $K^{\langle\sigma^2\rangle}$.

Observe that σ^2 carries

$$\omega \mapsto \omega^2 \mapsto \omega^4 \mapsto \omega$$
$$\omega^3 \mapsto \omega^6 \mapsto \omega^5 \mapsto \omega^3$$

Let $\gamma = \omega + \omega^2 + \omega^4$. Observe that $\sigma(\gamma) = \omega^3 + \omega^6 + \omega^5$. I bet that $(x-\gamma)(x-\sigma(\gamma))$ is the minimal polynomial of γ over $\mathbb{Q}$:

$$\begin{aligned}(x-\gamma)(x-\sigma(\gamma)) &= x^2 - \underbrace{(\gamma+\sigma(\gamma))}_{-1} x + \underbrace{(\gamma)\sigma(\gamma)}_{2} \\ &= x^2 + x + 2.\end{aligned}$$

(We calculate that $x^2 - x + 2$ is indeed irreducible over $\mathbb{Q}$ and is the minimal polynomial for both γ and $\sigma(\gamma)$.) Thus, $K^{\langle\sigma^2\rangle} = \mathbb{Q}[\gamma]$ and every element of $\operatorname{Aut}_{\mathbb{Q}} K$ restricts to give an element of $\operatorname{Aut}_{\mathbb{Q}} K^{\langle\sigma^2\rangle}$.

Goal. 5.28

(a) *If J is a finite group of automorphisms of the field K, then* $\dim_{K^J} K = |J|$.

(b) *If $\boldsymbol{k} \subseteq K$ is a finite dimensional Galois extension and E is an intermediate field, then*

$$\frac{|\operatorname{Aut}_{\boldsymbol{k}} K|}{|\operatorname{Aut}_E K|} = \dim_{\boldsymbol{k}} E.$$

We will show the following facts.

- (5.23) If $\boldsymbol{k} \subseteq K$ is a finite dimensional field extension, then $|\operatorname{Aut}_{\boldsymbol{k}} K| \le \dim_{\boldsymbol{k}} K$.
- (5.25) If $\boldsymbol{k} \subseteq E \subseteq K$ is a finite dimensional field extension, then $\frac{|\operatorname{Aut}_{\boldsymbol{k}} K|}{|\operatorname{Aut}_E K|} \le \dim_{\boldsymbol{k}} E$
- (5.26) If J is a finite group of automorphisms of a field K, then $\dim_{K^J} K \le |J|$.
- (5.27) If $H \subseteq J$ are finite groups of automorphisms of a field K, then $\dim_{K^J} K^H \le \frac{|J|}{|H|}$.

Lemma 5.23. *If $\boldsymbol{k} \subseteq K$ is a finite dimensional field extension, then* $|\operatorname{Aut}_{\boldsymbol{k}} K| \le \dim_{\boldsymbol{k}} K$.

Actually, we prove a stronger statement.

Stronger Statement 5.24. *Let $\boldsymbol{k}$, $\boldsymbol{k}'$, K, and K' be fields with $\boldsymbol{k} \subseteq K$, $\boldsymbol{k}' \subseteq K'$, and* $\dim_{\boldsymbol{k}} K$ *finite. If $\sigma : \boldsymbol{k} \to \boldsymbol{k}'$ is an isomorphism, then*

$$|\{\widetilde{\sigma} : K \to K' \mid \widetilde{\sigma} \textit{ is a field homomorphism and } \widetilde{\sigma}|_{\boldsymbol{k}} = \sigma\}| \le \dim_{\boldsymbol{k}} K.$$

It is clear that 5.24 implies 5.23, just take σ to be the identity map and $K' = K$. We prove 5.24.

Proof. We induct on $\dim_{\boldsymbol{k}} K$. Take $u \in K \setminus \boldsymbol{k}$. Let $f \in \boldsymbol{k}[x]$ be the minimum polynomial of u. Let f^σ be the image of f in $\boldsymbol{k}'[x]$ under σ. (In other words, if $f = \sum_i a_i x^i$, then $f^\sigma = \sum_i \sigma(a_i) x^i$.) Observe that if $\sigma_1 : \boldsymbol{k}[u] \to K'$ is an extension of σ, then $\sigma_1(u)$ is a root of f^σ. So there are at most $\dim_{\boldsymbol{k}} \boldsymbol{k}[u]$ extensions of σ to $\boldsymbol{k}[u]$. Once σ_1 is fixed, then, by induction, there are at most $\dim_{\boldsymbol{k}[u]} K$ extensions of σ_1 to K. Thus, altogether there are at most

$$\dim_{\boldsymbol{k}[u]} K \cdot \dim_{\boldsymbol{k}} \boldsymbol{k}[u] = \dim_{\boldsymbol{k}} K$$

extensions of σ to K. □

Lemma 5.25. *Let $\boldsymbol{k} \subseteq E \subseteq K$ be finite dimensional field extensions. Then*

$$\frac{|\operatorname{Aut}_{\boldsymbol{k}} K|}{|\operatorname{Aut}_E K|} \le \dim_{\boldsymbol{k}} E.$$

Proof. Partition $\operatorname{Aut}_{\boldsymbol{k}} K$ into subsets $S_1 \cup S_2 \cup \cdots \cup S_N$, for some integer N, where two elements σ and τ of $\operatorname{Aut}_{\boldsymbol{k}} K$ are in the same set S_i if and only if $\sigma|_E = \tau|_E$.

Observe that

$$N \le |\{\sigma : E \to K \mid \sigma|_{\boldsymbol{k}} = \mathrm{id}\}| \le \dim_{\boldsymbol{k}} E.$$

The inequality is a consequence of 5.24.

We next show that each S_i satisfies $|S_i| \le |\operatorname{Aut}_E K|$. Fix S_i and fix $\tau \in S_i$. Observe that if $\sigma \in S_i$, then $\sigma \circ \tau^{-1}$ is an element of $\operatorname{Aut}_{\boldsymbol{k}} K$ which acts like the identity map on E. Thus, $\sigma \circ \tau^{-1}$ is an element of $\operatorname{Aut}_E K$. Furthermore, if σ and σ' are distinct elements of S_i, then $\sigma \circ \tau^{-1}$ and $\sigma' \circ \tau^{-1}$ are distinct elements of $\operatorname{Aut}_E K$.

Thus

$$|\operatorname{Aut}_{\boldsymbol{k}} K| = \sum_{i=1}^{N} |S_i| \le \dim_{\boldsymbol{k}} E \cdot |\operatorname{Aut}_E K|$$

and

$$\frac{|\operatorname{Aut}_{\boldsymbol{k}} K|}{|\operatorname{Aut}_E K|} \le \dim_{\boldsymbol{k}} E.$$

□

Lemma 5.26. *If K is a field and J is a finite group of automorphisms of K, then* $\dim_{K^J} K \le |J|$.

Proof. Let $J = \{\sigma_1, \dots, \sigma_n\}$. Let $u_1, \dots, u_m$ be elements of K with $n < m$. We prove that $u_1, \dots, u_m$ are linearly dependent over K^J. Consider the n homogeneous linear equations in m unknowns

$$\begin{gathered}
\sigma_1(u_1)X_1 + \cdots + \sigma_1(u_m)X_m = 0\\
\sigma_2(u_1)X_1 + \cdots + \sigma_2(u_m)X_m = 0\\
\vdots\\
\sigma_n(u_1)X_1 + \cdots + \sigma_n(u_m)X_m = 0.
\end{gathered}$$

There is a non-trivial solution in K^m of this system of equations. We will show that there also is a non-trivial solution of this system of equations in $(K^J)^m$.[45] Let $\theta = \begin{bmatrix} a_1 & \cdots & a_m \end{bmatrix}^{\mathrm{T}}$ be a non-trivial solution of the system with the least number of non-zero entries. We also insist that one entry of θ is 1. Observe that $\sigma(\theta)$ is a solution of the system for all $\sigma \in J$. Also $\theta - \sigma(\theta)$ is a solution of the system with at least one more zero. Thus, $\theta - \sigma(\theta) = 0$ for all $\sigma \in J$. Thus, $\theta \in (K^J)^m$ and the proof is complete. □

The next result is the exact same set of tricks; but the statement has been jazzed up a little.

[45] Of course, we only promised to find a solution in $(K^J)^m$ of the one equation $u_1X_1 + \cdots + u_mX_m = 0$. Sometimes it is easier to do much more than you want in order to obtain what you want. The advantage of solving the entire system of equations (rather than trying to solve only the one equation) is that if some vector $\begin{bmatrix} a_1 & \cdots & a_m \end{bmatrix}^{\mathrm{T}}$ is a solution of the system, then $\begin{bmatrix} \sigma(a_1) & \cdots & \sigma(a_m) \end{bmatrix}^{\mathrm{T}}$ also is a solution of the system for all $\sigma \in J$.

Lemma 5.27. *Let K be a field and let $H \subseteq J$ be finite groups of automorphisms of K. Then*

$$\dim_{K^J} K^H \le \frac{|J|}{|H|}.$$

Proof. Select a full set of representatives $\sigma_1, \dots, \sigma_n$ of the cosets of H in J. (In other words, J is the disjoint union $\sigma_1 H \cup \cdots \cup \sigma_n H$.) Suppose $u_1, \dots, u_m$ are in K^H for some m with $n < m$.

We prove that $u_1, \dots, u_m$ are linearly dependent over K^J. Consider the n homogeneous linear equations in m unknowns

$$\begin{array}{c} \sigma_1(u_1)X_1 + \cdots + \sigma_1(u_m)X_m = 0 \\ \sigma_2(u_1)X_1 + \cdots + \sigma_2(u_m)X_m = 0 \\ \vdots \\ \sigma_n(u_1)X_1 + \cdots + \sigma_n(u_m)X_m = 0. \end{array}$$

There is a non-trivial solution in $(K^H)^m$ of this system of equations. We will show that there also is a non-trivial solution of this system of equations in $(K^J)^m$. [46] Let $\theta = \begin{bmatrix} a_1 & \cdots & a_m \end{bmatrix}^{\mathrm{T}} \in (K^H)^m$ be a non-trivial solution of the system with the least number of non-zero entries. We also insist that one entry of θ is 1. Observe that $\sigma(\theta)$ is a solution of the system for all $\sigma \in J$ because

$$\sigma \circ \sigma_1, \dots, \sigma \circ \sigma_n$$

is also a complete set of representatives of the cosets of H in J. Also $\theta - \sigma(\theta)$ is a solution of the system with at least one more zero. Thus, $\theta - \sigma(\theta) = 0$ for all $\sigma \in J$. Thus, $\theta \in (K^J)^m$ and the proof is complete. □

We have shown the following facts.

- (5.23) If $\boldsymbol{k} \subseteq K$ is a finite dimensional field extension, then $|\operatorname{Aut}_{\boldsymbol{k}} K| \le \dim_{\boldsymbol{k}} K$.
- (5.25) If $\boldsymbol{k} \subseteq E \subseteq K$ is a finite dimensional field extension, then $\frac{|\operatorname{Aut}_{\boldsymbol{k}} K|}{|\operatorname{Aut}_E K|} \le \dim_{\boldsymbol{k}} E$
- (5.26) If J is a finite group of automorphisms of a field K, then $\dim_{K^J} K \le |J|$.
- (5.27) If $H \subseteq J$ are finite groups of automorphisms of a field K, then $\dim_{K^J} K^H \le \frac{|J|}{|H|}$.

Corollary 5.28.

(a) *If J is a finite group of automorphisms of the field K, then* $\dim_{K^J} K = |J|$.

(b) *If $\boldsymbol{k} \subseteq K$ is a finite dimensional Galois extension and E is an intermediate field, then*

$$\frac{|\operatorname{Aut}_{\boldsymbol{k}} K|}{|\operatorname{Aut}_E K|} = \dim_{\boldsymbol{k}} E.$$

Proof. We first prove (a). Observe that

$$|J| \le |\operatorname{Aut}_{K^J} K| \le \dim_{K^J} K \le |J|.$$

The first inequality holds because $J \subseteq \operatorname{Aut}_{K^J} K$. The second inequality is Lemma 5.23. The final inequality is Lemma 5.26.

(b). Observe that

$$\dim_{\boldsymbol{k}} E \le \dim_{\boldsymbol{k}} K^{\operatorname{Aut}_E K} = \dim_{K^{\operatorname{Aut}_{\boldsymbol{k}} K}} K^{\operatorname{Aut}_E K} \le \frac{|\operatorname{Aut}_{\boldsymbol{k}} K|}{|\operatorname{Aut}_E K|} \le \dim_{\boldsymbol{k}} E.$$

[46] Once again, we prove more than we want. It still is true that if some vector $\begin{bmatrix} a_1 & \cdots & a_m \end{bmatrix}^{\mathrm{T}}$ is a solution of the system, then $\begin{bmatrix} \sigma(a_1) & \cdots & \sigma(a_m) \end{bmatrix}^{\mathrm{T}}$ also is a solution of the system for all $\sigma \in J$.

The left most inequality holds because $E \subseteq K^{\operatorname{Aut}_E K}$.

The equality holds because the extension $k \subseteq K$ is Galois by hypothesis; hence $\boldsymbol{k} = K^{\operatorname{Aut}_{\boldsymbol{k}} K}$.

The second inequality is due to Lemma 5.27 which says

$$H \subseteq J \Rightarrow \dim_{K^J} K^H \leq \frac{|J|}{|H|}.$$

The right-most inequality is Lemma 5.25.

□

The proof of the Fundamental Theorem of Galois Theory.

Let $\boldsymbol{k} \subseteq K$ be a finite dimensional Galois extension and let $G = \operatorname{Aut}_{\boldsymbol{k}} K$. The following statements hold.

- (a) There is a one-to-one (inclusion reversing) correspondence between the subgroups of G and the fields E with $\boldsymbol{k} \subseteq E \subseteq K$. The correspondence sends
$$H \longrightarrow K^H$$
$$\operatorname{Aut}_E K \longleftarrow E$$
- (b) If H is a subgroup of G, then $|H| = \dim_{K^H} K$.
- (c) The field K is Galois over each intermediate field E, with $\boldsymbol{k} \subseteq E \subseteq K$.
- (d) If E is an intermediate field, then
$$E \text{ is Galois over } \boldsymbol{k} \iff \operatorname{Aut}_E K \lhd \operatorname{Aut}_{\boldsymbol{k}} K.$$
Furthermore, in this case,
$$\operatorname{Aut}_{\boldsymbol{k}} E \cong \frac{\operatorname{Aut}_{\boldsymbol{k}} K}{\operatorname{Aut}_E K}.$$

(b). We proved this in 5.28.(a).

(a). For each subgroup H of G, let $\bar{H} = \operatorname{Aut}_{K^H} K$. For each intermediate field E, let $\bar{E} = K^{\operatorname{Aut}_E K}$. It suffices to prove $H = \bar{H}$ for all H and $E = \bar{E}$ for all E. It is clear that $H \subseteq \bar{H}$ (because H fixes all of the stuff fixed by H) and $E \subseteq \bar{E}$ (because E is fixed by the stuff that fixes E). It suffices to show that $|\bar{H}| = |H|$ and $\dim_{\boldsymbol{k}} E = \dim_{\boldsymbol{k}} \bar{E}$ for each subgroup H of G and for each intermediate field E with $\boldsymbol{k} \subseteq E \subseteq K$.

Observe that

$$K^H = K^{\bar{H}}. \tag{5.28.1}$$

Indeed,

$$H \subseteq \bar{H} \Rightarrow K^{\bar{H}} \subseteq K^H,$$

and the definition of $\bar{H}$ ensures that K^H is fixed by $\bar{H}$ (indeed, $\bar{H}$ is the set of automorphisms of K which fix K^H); consequently,

$$K^H \subseteq K^{\bar{H}}.$$

Thus, we apply 5.20.(b) twice (these are the outer equalities) to see that

$$|H| = \dim_{K^H} K = \dim_{K^{\bar{H}}} K = |\bar{H}|.$$

The middle equality is (5.28.1).

Similarly, observe that

$$\operatorname{Aut}_E K = \operatorname{Aut}_{\bar{E}} K. \tag{5.28.2}$$

Indeed, $E \subset \bar{E}$; hence, $\operatorname{Aut}_{\bar{E}} K \subseteq \operatorname{Aut}_E K$. On the other hand, the definition of $\bar{E}$ ensures that $\operatorname{Aut}_E K \subseteq \operatorname{Aut}_{\bar{E}} K$. Thus,

$$\dim_{\boldsymbol{k}} E = \frac{|\operatorname{Aut}_{\boldsymbol{k}} K|}{|\operatorname{Aut}_E K|} = \frac{|\operatorname{Aut}_{\boldsymbol{k}} K|}{|\operatorname{Aut}_{\bar{E}} K|} = \dim_{\boldsymbol{k}} \bar{E}.$$

The outer equalities are 5.28.(b) and the middle equality is 5.28.2.

(c). If E is an intermediate field, then E is $E = K^H$ for some H by (a); hence $E \subseteq K$ is a Galois extension by Remark 5.15. (One could also appeal directly to the proof of (a).)

We use a little Lemma in our proof of (d).

Lemma 5.29. *Let $\boldsymbol{k} \subseteq E$ be a finite dimensional Galois extension; let $u \in E$ and $f \in \boldsymbol{k}[x]$ be the minimal polynomial of u over $\boldsymbol{k}$. Then f factors into distinct linear factors in $E[x]$.*

Proof. Let $e_1, \dots, e_r$ be the roots of f in E. (List each root exactly once.) If $\sigma \in \operatorname{Aut}_{\boldsymbol{k}} E$, then σ permutes $e_1, \dots, e_r$; consequently, σ leaves every symmetric polynomial in $e_1, \dots, e_r$ fixed. In particular, σ leaves the coefficients of

$$g(x) = (x - e_1)(x - e_2) \cdots (x - e_r) = x^r - (e_1 + \cdots + e_r)x^{r-1} + \cdots + (-1)^r(e_1 e_2 \cdots e_r)$$

fixed for all $\sigma \in \operatorname{Aut}_{\boldsymbol{k}} E$. Thus, the $g \in \boldsymbol{k}[x]$, g has degree at most $\deg f$, and g is divisible by f. We conclude that $g = f$. We have already demonstrated that g splits into distinct linear factors in $E[x]$. □

Proof of Theorem 5.20.(d). $(\Rightarrow)$ We assume $\boldsymbol{k} \subseteq E \subseteq K$ are finite dimensional field extensions with $\boldsymbol{k} \subseteq K$ and $\boldsymbol{k} \subseteq E$ both Galois extensions. We prove that $\operatorname{Aut}_E K \lhd \operatorname{Aut}_{\boldsymbol{k}} K$ and

$$\frac{\operatorname{Aut}_{\boldsymbol{k}} K}{\operatorname{Aut}_E K} \cong \operatorname{Aut}_{\boldsymbol{k}} E.$$

The key idea is that every element of $\operatorname{Aut}_{\boldsymbol{k}} K$ restricts to be an automorphism of E. Let σ be an element of $\operatorname{Aut}_{\boldsymbol{k}} K$. If u is an element of E, then $\sigma(u)$ is a root of the minimal polynomial f of u over $\boldsymbol{k}$. We proved in Lemma 5.29 that all of the roots of f in K **are already in** E. Thus, $\sigma(E) \subseteq E$. The fields $\sigma(E) \subseteq E$ have the same dimension as vector spaces over $\boldsymbol{k}$; hence, the restriction of σ to E (denoted $\sigma|_E$) is an automorphism of E. Consider the group homomorphism

$$\operatorname{Aut}_{\boldsymbol{k}} K \to \operatorname{Aut}_{\boldsymbol{k}} E,$$

which is given by $\sigma \mapsto \sigma|_E$. The kernel of the map is $\operatorname{Aut}_E K$. The First Isomorphism Theorem ensures that

$$\frac{\operatorname{Aut}_{\boldsymbol{k}} K}{\operatorname{Aut}_E K}$$

is isomorphic to a subgroup of $\operatorname{Aut}_{\boldsymbol{k}} E$. On the other hand, the groups

$$\frac{\operatorname{Aut}_{\boldsymbol{k}} K}{\operatorname{Aut}_E K} \quad \text{and} \quad \operatorname{Aut}_{\boldsymbol{k}} E$$

both have order $\dim_{\boldsymbol{k}} E$ because

$$\boldsymbol{k} \subseteq E, \quad \boldsymbol{k} \subseteq K, \quad \text{and} \quad E \subseteq K$$

all are Galois extensions and we may use the earlier parts of the Theorem. Thus,

$$\frac{\operatorname{Aut}_{\boldsymbol{k}} K}{\operatorname{Aut}_E K}$$

is isomorphic to all of $\operatorname{Aut}_{\boldsymbol{k}} E$.

Proof of Theorem 5.20.(d). ($\Leftarrow$) We assume $\boldsymbol{k} \subseteq E \subseteq K$ are finite dimensional field extensions with $E \subseteq K$ a Galois extension and

$$\operatorname{Aut}_E K \lhd \operatorname{Aut}_{\boldsymbol{k}} K.$$

We prove that $\boldsymbol{k} \subseteq E$ is a Galois extension.

We first prove that every element of $\operatorname{Aut}_{\boldsymbol{k}} K$ restricts to an element of $\operatorname{Aut}_E K$.

Observe that

$$\sigma^{-1} \circ \tau \circ \sigma \in \operatorname{Aut}_E K, \quad \forall \sigma \in \operatorname{Aut}_{\boldsymbol{k}} K, \quad \forall \tau \in \operatorname{Aut}_E K,$$

$$(\sigma^{-1} \circ \tau \circ \sigma)(e) = e, \quad \forall \sigma \in \operatorname{Aut}_{\boldsymbol{k}} K, \quad \forall \tau \in \operatorname{Aut}_E K, \quad \forall e \in E$$

$$\tau(\sigma(e)) = \sigma(e), \quad \forall \sigma \in \operatorname{Aut}_{\boldsymbol{k}} K, \quad \forall \tau \in \operatorname{Aut}_E K, \quad \forall e \in E$$

$$\sigma(E) \subseteq K^{\operatorname{Aut}_E K} = E, \quad \forall \sigma \in \operatorname{Aut}_E K.$$

Once again, a $\dim_{\boldsymbol{k}}$-count[47] ensures that $\sigma|_E$ is in $\operatorname{Aut}_{\boldsymbol{k}} E$ for all $\sigma \in \operatorname{Aut}_E K$.

Once again we consider the group homomorphism

$$\operatorname{Aut}_{\boldsymbol{k}} K \to \operatorname{Aut}_{\boldsymbol{k}} E,$$

which is given by $\sigma \mapsto \sigma|_E$. The kernel still is $\operatorname{Aut}_E K$. Thus, once again,

$$\frac{\operatorname{Aut}_{\boldsymbol{k}} K}{\operatorname{Aut}_E K}$$

is isomorphic to a subgroup of $\operatorname{Aut}_{\boldsymbol{k}} E$ and

(5.29.1) $$\dim_{\boldsymbol{k}} E = \left| \frac{\operatorname{Aut}_{\boldsymbol{k}} K}{\operatorname{Aut}_E K} \right| \leq |\operatorname{Aut}_{\boldsymbol{k}} E|$$

Consider the fields

(5.29.2) $$\boldsymbol{k} \subseteq E^{\operatorname{Aut}_{\boldsymbol{k}} E} \subset E.$$

Observe that

$$\dim_{\boldsymbol{k}} E \leq |\operatorname{Aut}_{\boldsymbol{k}} E| = \dim_{E^{\operatorname{Aut}_{\boldsymbol{k}} E}} E \leq \Big(\dim_{E^{\operatorname{Aut}_{\boldsymbol{k}} E}} E \Big) \Big(\dim_{\boldsymbol{k}} E^{\operatorname{Aut}_{\boldsymbol{k}} E} \Big) = \dim_{\boldsymbol{k}} E.$$

(The left most inequality is (5.29.1); the equality holds because $E^{\operatorname{Aut}_{\boldsymbol{k}} E} \subseteq E$ is a Galois extension.) Equality holds across the board and $\dim_{\boldsymbol{k}} E^{\operatorname{Aut}_{\boldsymbol{k}} E} = 1$. Thus $\boldsymbol{k} = E^{\operatorname{Aut}_{\boldsymbol{k}} E}$ and $\boldsymbol{k} \subseteq E$ is a Galois extension.

[47] We have three fields $\boldsymbol{k} \subseteq \sigma(E) \subseteq E$ and we know that $\dim_{\boldsymbol{k}} \sigma(E) = \dim_{\boldsymbol{k}} E$. Hence we conclude $\sigma(E) = E$.

5.D. Three ways to say "the field extension $\boldsymbol{k} \subseteq K$ is a Galois extension".

Theorem 5.30. *If $\boldsymbol{k} \subseteq K$ is a finite dimensional field extension, then the following statements are equivalent.*

(a) [48] *There exists a finite group of automorphisms G of K such that $K^G = \boldsymbol{k}$.*
(b) *If $u \in K$ and $f \in \boldsymbol{k}[x]$ is the minimal polynomial of u over $\boldsymbol{k}$, then f factors into distinct linear factors in $K[x]$.*
(c) *There is a polynomial $f \in \boldsymbol{k}[x]$ such that $f = \prod_{i=1}^{n}(x - u_i)$ in $K[x]$, with all u_i distinct and $K = \boldsymbol{k}[u_1, \dots, u_n]$.*

Proof. **(a)⇒(b)** See Lemma 5.29.

(b)⇒(c) Let $K = \boldsymbol{k}[u_1, \dots, u_n]$ and let f_i be the minimal polynomial of u_i over $\boldsymbol{k}$. According to hypothesis (b), each f_i splits into a product of distinct linear factors in $K[x]$. Toss away any repeats among the f_i. Multiply the remaining f_i together; call this product f. Observe that f splits into a product of distinct linear factors in $K[x]$ and K is obtained from $\boldsymbol{k}$ by adjoining all of the roots of f.

(c)⇒(a) We prove that there are at least $\dim_{\boldsymbol{k}} K$ distinct elements of $\operatorname{Aut}_{\boldsymbol{k}} K$. Once we do this, then we consider the fields

$$\boldsymbol{k} \subseteq K^{\operatorname{Aut}_{\boldsymbol{k}} K} \subseteq K.$$

Observe that

$$K^{\operatorname{Aut}_{\boldsymbol{k}} K} \subseteq K$$

is a Galois extension with

$$\dim_{K^{\operatorname{Aut}_{\boldsymbol{k}} K}} K \leq \dim_{K^{\operatorname{Aut}_{\boldsymbol{k}} K}} K \cdot \dim_{\boldsymbol{k}} K^{\operatorname{Aut}_{\boldsymbol{k}} K} = \dim_{\boldsymbol{k}} K \leq |\operatorname{Aut}_{\boldsymbol{k}} K| = \dim_{K^{\operatorname{Aut}_{\boldsymbol{k}} K}} K.$$

(The inequality holds because $1 \leq \dim_{\boldsymbol{k}} K^{\operatorname{Aut}_{\boldsymbol{k}} K}$; the left-most equality is Observation 5.3; the middle equality holds because we plan to exhibit $\dim_{\boldsymbol{k}} K$ distinct elements of $\operatorname{Aut}_{\boldsymbol{k}} K$; the right most equality holds because $K^{\operatorname{Aut}_{\boldsymbol{k}}} K \subseteq K$ is a Galois extension and we may apply Theorem 5.20.) Hence, equality occurs across the board and $\dim_{\boldsymbol{k}} K^{\operatorname{Aut}_{\boldsymbol{k}} K} = 1$; that is $\boldsymbol{k} = K^{\operatorname{Aut}_{\boldsymbol{k}} K}$ and $\boldsymbol{k} \subseteq K$ is a Galois extension.

Here is how we produce[49] a large number of distinct elements of $\operatorname{Aut}_{\boldsymbol{k}} K$. For each i and each $\sigma : \boldsymbol{k}[u_1, \dots, u_{i-1}] \to K$, we prove that there exist

$$\dim_{\boldsymbol{k}[u_1,\dots,u_{i-1}]} \boldsymbol{k}[u_1, \dots, u_i]$$

distinct extensions of σ to $\boldsymbol{k}[u_1, \dots, u_i]$.

Let $E = \boldsymbol{k}[u_1, \dots, u_{i-1}]$; $\sigma : E \to K$ be a fixed ring homomorphism which fixes $\boldsymbol{k}$; and E' be $\sigma(E)$. Let g in $E[x]$ be the minimal polynomial of u_i over E. Of course, f is also in $E[x]$ and $f(u_i) = 0$. It follows that $f \in (g)E[x]$ and $f = gg_1$ for some g_1 in $E[x]$. Observe that

$$f = f^\sigma = g^\sigma g_2^\sigma.$$

[48] The field extension $\boldsymbol{k} \subseteq K$ is finite dimensional; consequently, the hypothesis of (a) is equivalent to $\boldsymbol{k} \subseteq K$ is a Galois extension; see Remark 5.15.

[49] This is an important argument. We use it again in the proof of 5.35 on page 110 and in the proof of 5.51 on page 115.

(We have used this notation before, if $h = \sum a_i x^i$ is in $E[x]$, then $h^\sigma = \sum \sigma(a_i)x^i \in E'[x]$. Keep in mind that the coefficients of f are in $\boldsymbol{k}$ and σ acts like the identity map on $\boldsymbol{k}$.) At any rate, g^σ splits into distinct linear factors in $K[x]$. Of course the isomorphism $\sigma : E \to E'$ induces an isomorphism

$$\frac{E[x]}{(g)} \to \frac{E'[x]}{(g^\sigma)}.$$

There are degree g distinct extensions of σ to $E[u_i]$; namely,

$$E[u_i] \to \frac{E[x]}{(g)} \to \frac{E'[x]}{(g^\sigma)} \to E'[\text{root}],$$

with

$$u_i \mapsto x \mapsto x \mapsto \text{root}.$$

There is such an extension for each of the degree g roots of g^σ in K. □

Example 5.31. Theorem 5.30 provides a quick answer to the questions

- Is $\mathbb{Q} \subseteq \mathbb{Q}[\sqrt[3]{2}]$ a Galois extension?
- Is $\mathbb{Q} \subseteq \mathbb{Q}[\omega]$ a Galois extension, where $\omega = e^{2\pi i/7}$?

We asked these questions in 5.17. Our answers, especially for ω, were fairly long winded. At this point our answer for $\mathbb{Q} \subseteq \mathbb{Q}[\sqrt[3]{2}]$ is: no, because $\sqrt[3]{2}$ is in $\mathbb{Q}[\sqrt[3]{2}]$ and the minimal polynomial for $\sqrt[3]{2}$ over $\mathbb{Q}$ does not split into distinct linear factors in $(\mathbb{Q}[\sqrt[3]{2}])[x]$. Our answer for $\mathbb{Q} \subseteq \mathbb{Q}[\omega]$ is the very quick, yes, because $\mathbb{Q}[\omega]$ is obtained by adjoining all of the roots $\frac{x^7-1}{x-1}$ to $\mathbb{Q}$ and this polynomial has distinct roots in $\mathbb{Q}[\omega]$.

We next write Theorem 5.30 in a slightly different language. First, we introduce the language.

Definition 5.32. The field extension $\boldsymbol{k} \subseteq K$ is called normal if for every $u \in K$, the minimal polynomial of u over $\boldsymbol{k}$ factors into linear factors in $K[x]$.

Definition 5.33. The polynomial $f \in \boldsymbol{k}[x]$ is called separable if every irreducible factor of f factors into distinct linear factors in some normal extension K of $\boldsymbol{k}$. The field extension $\boldsymbol{k} \subseteq K$ is separable if the minimal polynomial of every element of K over $\boldsymbol{k}$ is a separable polynomial.

Definition 5.34. If $f \in \boldsymbol{k}[x]$, then the field K is called the splitting field of f if $f = \prod_{i=1}^n (x - u_i)$ in $K[x]$ and $K = \boldsymbol{k}[u_1, \dots, u_n]$.

Theorem. [This is a reformulation of Theorem 5.30] *Let $\boldsymbol{k} \subseteq K$ be a finite dimensional field extension. Then the following statements are equivalent*

(a) *$\boldsymbol{k} \subseteq K$ is a Galois extension;*
(b) *$\boldsymbol{k} \subseteq K$ is a separable normal field extension; and*
(c) *K is the splitting field of some separable polynomial f in $\boldsymbol{k}[x]$.*

5.D.a. *Splitting fields.*

Theorem 5.35. *Let $\boldsymbol{k}$ be a field and f be a polynomial in $\boldsymbol{k}[x]$. The following statements hold.*

(a) *There exists a field K such that K is the splitting field of f over $\boldsymbol{k}$.*

(b) *If K and K' both are splitting fields of f over $\boldsymbol{k}$, then there is a $\boldsymbol{k}$-algebra isomorphism of K and K'.*

Proof. **(a)** It suffices to show that there exists a field E such that $E = \boldsymbol{k}[\alpha]$ and $f(\alpha) = 0$. Of course, this is obvious. Let g be an irreducible factor of f with $g(\alpha) = 0$ and take

$$E = \frac{\boldsymbol{k}[x]}{(g)}.$$

(b) Consider the partially order set $\{(E, \sigma)\}$ where $\boldsymbol{k} \subseteq E \subseteq K$ and $\sigma : E \to K'$ is a $\boldsymbol{k}$-algebra homomorphism. The order on this partially order set is given by

$$(E, \sigma) \leq (E', \sigma') \iff E \subseteq E' \quad \text{and} \quad \sigma'|E = \sigma.$$

Pick (E, σ) in this poset with $\dim_{\boldsymbol{k}} E$ maximal. We claim that

(i) $E = K$, and

(ii) $\sigma(E) = K'$.

Assertion (i) is obvious. If $E \neq K$, then we can extend σ using the technique of the proof of (c)⇒(a) in Theorem 5.30. Assertion (ii) is also fairly clear. The field K is equal to $K = \boldsymbol{k}[u_1, \dots, u_n]$, where $f = \prod(x - u_i)$ in K. It follows that $\sigma(K) = \boldsymbol{k}[\sigma(u_1), \dots, \sigma(u_n)]$, with

$$f = \prod(x - \sigma(u_i)) \in \sigma(K) \subseteq K'.$$

On the other hand, $K' = \boldsymbol{k}[\text{the roots of } f \text{ in } K']$. Thus, $K' = \boldsymbol{k}[\sigma(u_1), \dots, \sigma(u_n)] = \sigma(K)$. □

5.D.b. *Perfect fields.*

If R is a ring, then there is a ring homomorphism $\phi : \mathbb{Z} \to R$ with $\phi(1) = 1$. This ring homomorphism has a kernel which is generated by a non-negative integer. The non-negative integer which generates $\ker(\phi)$ is called the characteristic of R. If R is a field, then the characteristic of R is either 0 or some positive prime integer.

Definition 5.36. The field $\boldsymbol{k}$ is perfect if every polynomial $f \in \boldsymbol{k}[x]$ is separable.

In this section, we work out the Warm-up Examples of 5.37 and prove Theorem 5.38.

Warm-up Examples 5.37.

(a) Every polynomial in $\mathbb{Q}[x]$ is separable.

(b) Let $\boldsymbol{k}$ be a field of positive characteristic p, then $f = x^p - a$ in $\boldsymbol{k}[x]$ is separable if and only if f has a root in $\boldsymbol{k}$. Furthermore, if f has no root in $\boldsymbol{k}$, then f is irreducible, but not separable.

(c) If $\boldsymbol{k}$ is a field of positive characteristic p, then the polynomial $x^p - t$ is irreducible, but not separable, in $\boldsymbol{k}(t)[x]$.

Theorem 5.38. *If $\boldsymbol{k}$ is a field of characteristic zero or if $\boldsymbol{k}$ is a finite field, then $\boldsymbol{k}$ is a perfect field.*

Example 5.37.(a) Every polynomial in $\mathbb{Q}[x]$ is separable.

It suffices to show that if $f \in \mathbb{Q}[x]$ is irreducible, then f has distinct roots in $\mathbb{C}$. Suppose f is irreducible in $\mathbb{Q}[x]$ and $f = (x-\alpha)^2 g$ in $\mathbb{C}[x]$, for some $\alpha \in \mathbb{C}$ and some $g \in \mathbb{C}[x]$. Then $f' = 2(x-\alpha)g + (x-\alpha)^2 g'$. Thus, f' is in $\mathbb{Q}[x]$ and $f'(\alpha) = 0$. This is not possible because f' is in the ideal (f), $\deg f' < \deg f$, and f' is not the zero polynomial. □

Example 5.37.(b) Let $\boldsymbol{k}$ be a field of positive characteristic p, then $f = x^p - a$ in $\boldsymbol{k}[x]$ is separable if and only if f has a root in $\boldsymbol{k}$. Furthermore, if f has no root in $\boldsymbol{k}$, then f is irreducible, but not separable.

($\Leftarrow$) Suppose $b \in \boldsymbol{k}$ and $b^p = a$. Then $f = (x-b)^p$; the irreducible factors of f are all $x - b$ and $x - b$ is a separable polynomial. We conclude that every irreducible factor of $x^p - a$ is separable; hence, $x^p - a$ is a separable polynomial.

($\Rightarrow$) Suppose f a separable polynomial. In this case, there is a splitting field K for f and in $K[x]$, all of the irreducible factors f have distinct roots. Let $b \in K$ be one of the roots of $x^p - a$. Observe that in $K[x]$, $x^p - a = (x-b)^p$. Let $g \in \boldsymbol{k}[x]$ be an irreducible factor of f. Then $g = (x-b)^c$ for some c. The hypothesis that f is a separable polynomial guarantees that g has distinct roots in the splitting field of g; hence, g has distinct roots in K; thus $c = 1$ and b is in $\boldsymbol{k}$.

Now we prove the furthermore statement. We must show that if $x^p - a$ has no roots in $\boldsymbol{k}$, then $x^p - a$ is irreducible in $\boldsymbol{k}[x]$. Suppose $x^p - a$ has a non-trivial factor in $\boldsymbol{k}[x]$. This factor must look $(x-b)^r$ in $K[x]$ for some r with $1 \le r \le p-1$. Thus $b^r \in \boldsymbol{k}$. The integers r and p are relatively prime; so there exist integers ℓ and m with $\ell r + mp = 1$. Thus,

$$b = b^{\ell r + mp} = (b^r)^\ell (b^p)^m \in \boldsymbol{k}.$$

This is a contradiction. We conclude that if $x^p - a$ has no root in $\boldsymbol{k}$, then $x^p - a$ is irreducible in $\boldsymbol{k}[x]$. □

Example 5.37.(c) If $\boldsymbol{k}$ is a field of positive characteristic p, then the polynomial $x^p - t$ is irreducible, but not separable, in $\boldsymbol{k}(t)[x]$.

According to Example 5.37.(b), it suffices to prove that there does not exist $\alpha \in \boldsymbol{k}(t)$ with $\alpha^p = t$. An arbitrary element of $\boldsymbol{k}(t)$ looks like

$$\alpha = \frac{\sum a_i t^i}{\sum b_j t^j}$$

Observe that

$$t \neq \frac{\sum a_i^p t^{ip}}{\sum b_j^p t^{jp}}.$$

Indeed all of the exponents of

$$t \sum b_j^p t^{jp}$$

are congruent to one mod p and all of the exponents of

$$\sum a_i^p t^{ip}$$

are divisible by p. These two polynomials are equal only if each a_i and each b_j is zero. □

We now head in the direction of proving

Theorem. 5.38. *If $\boldsymbol{k}$ is a field of characteristic zero or if $\boldsymbol{k}$ is a finite field, then $\boldsymbol{k}$ is a perfect field.*

Step 5.39. *It makes sense to take the derivative in $\boldsymbol{k}[x]$ for all fields $\boldsymbol{k}$.*

Step 5.40. *If $\boldsymbol{k}$ is an arbitrary field and $f \in \boldsymbol{k}[x]$ is an irreducible polynomial, then f has repeated roots (in the splitting field E of f over $\boldsymbol{k}$) if and only if $f' = 0$.*

Corollary 5.41.

(a) *If $\boldsymbol{k}$ is a field of characteristic zero, then $\boldsymbol{k}$ is perfect.*

(b) *If $\boldsymbol{k}$ is a field of positive characteristic p, and $f \in \boldsymbol{k}[x]$ is an irreducible polynomial with repeated roots (in the splitting field E of f over $\boldsymbol{k}$), then $f(x) = a_0 + a_1x^p + \cdots + a_nx^{np}$.*

Step 5.42. *If $\boldsymbol{k}$ is a field of positive characteristic p, then $\boldsymbol{k}$ is perfect if and only if $\boldsymbol{k}^p = \boldsymbol{k}$.*

Step 5.43. *If $\boldsymbol{k}$ is a finite field, then $\boldsymbol{k}$ is perfect.*

We work on

Step. 5.39. *It makes sense to take the derivative in $\boldsymbol{k}[x]$ for all fields $\boldsymbol{k}$.*

Definition 5.44. Let $\boldsymbol{k}$ be a field. If $f(x) \in \boldsymbol{k}[x]$, then $f(x+h) \in \boldsymbol{k}[x,h]$ and

$$f(x+h) = f(x) + f_1(x)h + \cdots + f_n(x)h^n,$$

for some n and some $f_i \in \boldsymbol{k}[x]$. Define $f'(x) = f_1(x)$.

Observation 5.45. *If $\boldsymbol{k}$ is a field and f and g are elements of $k[x]$, then the following statements hold:*

(a) $(f+g)' = f' + g'$,

(b) $(x^n)' = nx^{n-1}$,

(c) $(a_0 + a_1x + \cdots + a_nx^n)' = a_1 + 2a_2x + \cdots + na_nx^{n-1}$, *and*

(d) $(fg)' = fg' + f'g$.

Proof. These are all easy. We will prove the product rule. Observe that

$$\begin{aligned} f(x+h)g(x+h) &= (f(x) + f_1(x)h + \dots)(g(x) + g_1(x)h + \dots) \\ &= f(x)g(x) + h\Big(f(x)g_1(x) + f_1(x)g(x)\Big) + h^2 \dots . \end{aligned}$$

Conclude that $(fg)' = fg' + f'g$, as claimed. □

Step. 5.40. *If $\boldsymbol{k}$ is an arbitrary field and $f \in \boldsymbol{k}[x]$ is an irreducible polynomial, then f has repeated roots (in the splitting field E of f over $\boldsymbol{k}$) if and only if $f' = 0$.*

Proof.
($\Rightarrow$) If f has repeated roots, then $f = (x-a)^r g(x)$ in $E[x]$, where E is a splitting field of f over $\boldsymbol{k}$. It follows that $f' = (x-a)^r g'(x) + r(x-a)^{r-1}g(x)$ in $E[x]$. Notice that f' is in $\boldsymbol{k}[x]$ and $f'(a) = 0$ in $\boldsymbol{k}$. Thus, $f' \in (f)\boldsymbol{k}[x]$; hence f' is identically zero.

($\Leftarrow$) Suppose $f(x) = \prod_{i=1}^{n}(x-a_i)^{e_i}$ in $E[x]$ where E is the splitting field of f over $\boldsymbol{k}$; the a_i are distinct; and the e_i are positive integers. Suppose also that $f' = 0$. Thus,

$$0 = \sum_j (x-a_1)^{e_1}\cdots e_j(x-a_j)^{e_j-1}\cdots(x-a_n)^{e_n}$$

$$= (\prod_{i=1}^{n}(x-a_i)^{e_i-1})\Big(e_1(x-a_2)\cdots(x-a_n)+e_2(x-a_1)(x-a_3)\cdots(x-a_n)+\cdots+e_n(x-a_1)\cdots(x-a_{n-1})$$

The polynomial ring $E[x]$ is a domain; consequently,

$$0 = e_1(x-a_2)\cdots(x-a_n)+e_2(x-a_1)(x-a_3)\cdots(x-a_n)+\cdots+e_n(x-a_1)\cdots(x-a_{n-1}).$$

Observe that each e_i is zero in E. Indeed, plug in a_1 to see that

$$0 = e_1(a_1-a_2)\cdots(a_1-a_n)$$

in E. We already insisted that the a's are distinct.

If $f' = 0$, then the integers $e_1, \dots, e_n$ are all zero in the field E. It follows that the characteristic of E (which is the same as the characteristic of $\boldsymbol{k}$) is a positive prime and this prime divides all of the exponents e_i. Notice that if the characteristic of $\boldsymbol{k}$ is the positive prime p, then $(x-a)^{pr} = (x^p - a^p)^r$ and this is a polynomial in the symbol x^p.

We conclude

- If $f' = 0$, then every root of f (in the splitting field of f over $\boldsymbol{k}$) is a repeated root.
- If the characteristic of $\boldsymbol{k}$ is zero, then $\boldsymbol{k}$ is a perfect field.
- If $\boldsymbol{k}$ is a field of positive characteristic p, and $f \in \boldsymbol{k}[x]$ is an irreducible polynomial with repeated roots (in the splitting field E of f over $\boldsymbol{k}$), then $f(x) = a_0 + a_1x^p + \cdots + a_nx^{np}$.

□

In particular, we have completely established Step 5.40 and Corollary 5.41.

The proof of step 5.42: If $\boldsymbol{k}$ is a field of positive characteristic p, then $\boldsymbol{k}$ is perfect if and only if $\boldsymbol{k}^p = \boldsymbol{k}$.

(Not $\Leftarrow$) Assume that there exists an element a in $\boldsymbol{k}$ with $a \neq b^p$ for any element $b \in \boldsymbol{k}$. Apply Example 5.37.(b) to see that $f = x^p - a$ is a non-separable irreducible polynomial in $\boldsymbol{k}[x]$. It follows that $\boldsymbol{k}$ is not a perfect field.

(Not $\Rightarrow$) Assume $\boldsymbol{k}$ is not a perfect field. Thus, there exists an irreducible non-separable polynomial f in $\boldsymbol{k}[x]$. According to Corollary 5.41.(b), f has the form $f(x) = a_0 + a_1x^p + \cdots + a_nx^{np}$. Observe that some a_i is NOT a p^{th} power of some element of $\boldsymbol{k}$ because if $a_i = b_i^p$ for some element $b_i \in \boldsymbol{k}$ for all i, then f would equal $(b_0 + b_1x + \cdots + b_nx^n)^p$ which is not irreducible. □

The proof of step 5.43: If $\boldsymbol{k}$ is a finite field, then $\boldsymbol{k}$ is perfect.

In light of 5.42 it suffices to prove that if $\boldsymbol{k}$ is a finite field of characteristic p, then $\boldsymbol{k}^p = \boldsymbol{k}$. Consider the field homomorphism

$$\phi : \boldsymbol{k} \to \boldsymbol{k}^p$$

which is given by $a \mapsto a^p$ for all $a \in \boldsymbol{k}$. (Of course, this is a homomorphism because $\boldsymbol{k}$ has characteristic p.) This field homomorphism, like all field homomorphisms, is injective. Thus, $\boldsymbol{k}^p$ is a subset of the finite set $\boldsymbol{k}$ and the two sets have the same number of elements. We conclude that $\boldsymbol{k} = \boldsymbol{k}^p$ □

5.E. **The field of complex numbers is algebraically closed.** The primitive element theorem is an elegant application of the Fundamental Theorem of Galois Theory.

Theorem 5.46. [The primitive element theorem] *If $\boldsymbol{k} \subseteq K$ is a field extension which is finite dimensional and separable, then $K = \boldsymbol{k}[u]$ for some $u \in K$.*

Proof. The vector space dimension $\dim_{\boldsymbol{k}} K$ is finite. It follows that the fields $\boldsymbol{k}$ and K are either both finite or both infinite. The proof has two cases.

Suppose first that $\boldsymbol{k}$ and K are both finite. In this case, $(K \setminus \{0\}, \times)$ is a cyclic group. (See, Remark (e) after Corollary 2.29, or Corollary 2.40, or Homework 12(c).) Let α be an element of K with the property that $\langle \alpha \rangle = (K \setminus \{0\}, \times)$. Observe that $K = \boldsymbol{k}[\alpha]$.

Henceforth, we assume that both fields $\boldsymbol{k}$ and K are infinite. The extension $\boldsymbol{k} \subseteq K$ is finite dimensional and separable; hence there exits a field $\widetilde{K}$ such that $K \subseteq \widetilde{K}$ and $\boldsymbol{k} \subseteq \widetilde{K}$ is a finite dimensional Galois extension.[50] The fact that $\boldsymbol{k} \subseteq \widetilde{K}$ is a finite dimensional Galois extension guarantees that there are only finitely many fields between $\boldsymbol{k}$ and $\widetilde{K}$. It follows that there are only finitely many fields between $\boldsymbol{k}$ and K. (We have no further use for $\widetilde{K}$.) Let u be an element in K with $\dim_{\boldsymbol{k}} \boldsymbol{k}[u]$ as large as possible. Of course, we claim that $\boldsymbol{k}[u] = K$. Suppose not. We will draw a contradiction. Suppose there exists $v \in K \setminus \boldsymbol{k}[u]$. Consider all fields of the form

$$\{\boldsymbol{k}[u + av] \mid a \in \boldsymbol{k}\}. \tag{5.46.1}$$

The set (5.46.1) is finite; but the index set $\boldsymbol{k}$ is infinite. Thus, there exist $a \neq b \in \boldsymbol{k}$ with

$$\boldsymbol{k}[u + av] = \boldsymbol{k}[u + bv].$$

Thus,

$$u + av \quad \text{and} \quad u + bv$$

are both elements $\boldsymbol{k}[u + av]$; and

$$(a - b)v = (u + av) - (u + bv) \in \boldsymbol{k}[u + av].$$

The element $a - b$ is a unit in $\boldsymbol{k}$. It follows that u and v are both in the field $\boldsymbol{k}[u + av]$. Thus,

$$\dim_{\boldsymbol{k}} \boldsymbol{k}[u] < \dim_{\boldsymbol{k}} \boldsymbol{k}[u + av];$$

which contradicts the choice of u. □

Theorem 5.47. *The field of complex numbers is algebraically closed.*

Proof. It suffices to prove that if K is a finite dimensional field extension of $\mathbb{C}$ with $\mathbb{R} \subset K$ a Galois extension, then $K = \mathbb{C}$.[51] Suppose $\dim_{\mathbb{R}} K = 2^m r$, with r odd. Let H be a Sylow subgroup of

[50] If $K = \boldsymbol{k}[u_1, \dots, u_n]$ and f_i is the minimal polynomial of u_i over $\boldsymbol{k}$, then take $\widetilde{K}$ to be a splitting field of $\prod_{i=1}^n f_i$ over K. In this case, $\widetilde{K}$ is also the splitting field of a separable polynomial over $\boldsymbol{k}$; and hence $\boldsymbol{k} \subseteq \widetilde{K}$ is a Galois extension; see the Theorem just under Definition 5.34.

[51] Indeed, any finite dimensional field extension of $\mathbb{C}$ can be extended to such a K; see footnote 50 on page 114.

$\operatorname{Aut}_{\mathbb{R}} K$ of order 2^m. Observe that $\mathbb{R} \subseteq K^H \subseteq K$ are fields with $\dim_{\mathbb{R}} K^H = r$ and $\dim_{K^H} K = 2^m$. Apply the Primitive Element Theorem (Theorem 5.46) to see that $K^H = \mathbb{R}[u]$ for some u. The minimal polynomial of u over $\mathbb{R}$ is an irreducible polynomial in $\mathbb{R}[x]$ of odd degree. Recall from calculus that every polynomial in $\mathbb{R}[x]$ of odd degree has a root.[52] If $f \in \mathbb{R}[x]$ is irreducible and has a real root, then f is a linear polynomial; hence, $r = 1$.

Thus $\mathbb{C} \subseteq K$ is a Galois extension of dimension 2^n for some n. If n is positive, then apply the Sylow Theorems again to obtain a subgroup H of $\operatorname{Aut}_{\mathbb{C}} K$ of order 2^{n-1}. Observe that $\mathbb{C} \subseteq K^H$ is a field extension of dimension 2. It follows that K^H is obtained by adjoining an element u to $\mathbb{C}$ with u^2 in $\mathbb{C}$.[53] On the other hand, $\mathbb{C}$ is closed under the taking of square roots;[54] Thus, there do not exist any two-dimensional field extensions of $\mathbb{C}$. We conclude that "n is not positive". In other words, $\mathbb{C} = K$ for all finite dimensional field extensions K of $\mathbb{C}$. □

5.F. **Every field has an algebraic closure.**

Definition 5.48. The field K is called algebraically closed if every polynomial $f \in K[x]$ of positive degree has a root in K. If $\boldsymbol{k} \subseteq K$ is a field extension then K is the algebraic closure of $\boldsymbol{k}$ if K is algebraically closed and K is algebraic over $\boldsymbol{k}$.

Example 5.49. The field $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$.

In this section, we prove two results.

Theorem 5.50. *If $\boldsymbol{k}$ is an arbitrary field, then there exists an algebraic closure K of $\boldsymbol{k}$.*

Theorem 5.51. *If $\boldsymbol{k}$ is an arbitrary field, and K_1 and K_2 are algebraic closures of $\boldsymbol{k}$, then K_1 and K_2 are isomorphic over*[55] *$\boldsymbol{k}$.*

We will use Zorn's Lemma twice to prove these Theorems. My guess is that you have used Zorn's Lemma already in Analysis and/or Topology. Zorn's Lemma is the way that most Mathematicians use the Axiom of Choice. The Axiom of Choice is a benign seeming statement which in fact is independent of the usual Zermelo-Fraenkel axioms of Set Theory. (This means that one can assume the Zermelo-Fraenkel axioms, assume the axiom of choice is false, and still have a coherent, consistent system of axioms in which to prove Mathematical Theorems.) Most Mathematicians assume that the Axiom of Choice holds. (Although most folks point out "I used the axiom of choice here;" and most folks avoid using the axiom of choice when this is possible.)

The axiom of choice states: For any set X of nonempty sets, there exists a function

$$f : X \to \cup_{A \in X} A$$

such that $f(A) \in A$ for all $A \in X$.

[52] If $f = x^r +$ l.o.t. and r is odd, then $\lim\limits_{x\to+\infty} f = +\infty$ and $\lim\limits_{x\to-\infty} f = -\infty$. Apply the Intermediate Value Theorem.

[53] I state and prove this Lemma carefully in my proof of 5c in "Low Lying Fruit".

[54] If $z = re^{i\theta}$ is an arbitrary element of $\mathbb{C}$, with r and θ in $\mathbb{R}$ and r non-negative, then $\sqrt{r}e^{i\theta/2}$ is an element of $\mathbb{C}$ which squares to z.

[55] The field $\boldsymbol{k}$ is a subfield of both K_1 and K_2. The assertion That K_1 and K_2 are isomorphic over $\boldsymbol{k}$ means that there is a field isomorphism $\phi : K_1 \to K_2$ with ϕ acting like the identity map on $\boldsymbol{k}$.

Two statements from Algebra which are each equivalent to the Axiom of choice are

- Every vector space has a basis.
- Every ring has a maximal ideal.

Zorn's Lemma is also equivalent to the axiom of choice. It is a formulation that leads to applications.

Zorn's Lemma. Let P be a non-empty partially ordered set. Assume that every chain in P has an upper bound in P. Then P contains at least one maximal element.

Definition 5.52. A set P together with a relation $\le$ on P is called a partially ordered set if the relation $\le$ satisfies:

- Reflexivity: If $a \in P$, then $a \le a$.
- Anti-symmetry: If $a, b \in P$ and $a \le b$ and $b \le a$, then $a = b$.
- Transitivity: If $a, b, c \in P$, with $a \le b$ and $b \le c$, then $a \le c$.

Definition 5.53. A chain in a partially ordered set is a totally order subset. That is, if a and b are elements of the chain then $a \le b$ or $b \le a$.

Definition 5.54. If S is a subset in a partially ordered set P, then an upper bound for S in P is an element $a \in P$ with $b \le a$ for all $b \in S$.

Definition 5.55. An element a in a partially ordered set P is a maximal element of P if there does not exist any element $b \in P$ with $a \le b$ other than $b = a$.

Here is an elementary application of Zorn's Lemma; we will appeal to this result when we prove that every field has an algebraic closure. In this application it is not necessary to assume that the ring R is commutative or Noetherian. (Of course, if R is commutative and Noetherian, then one need not use Zorn's Lemma.) It turns out that one can deduce Zorn's Lemma from the next result – although, that fact is not related to our purposes.

Theorem 5.56. *Let I be a proper two-sided ideal in a (not necessarily commutative) ring R. Then there exists a maximal two-sided ideal $\mathfrak{m}$ of R with $I \subseteq \mathfrak{m}$.*

Proof. We apply Zorn's Lemma. Let P be the set of proper two-sided ideals of R which contain I. Observe that P is non-empty because I is in P.[56] View P as a partially ordered set with relation $\subseteq$. Let $C = \{C_\ell | \ell \in L\}$ be a totally order subset of P. Observe that

$$J = \bigcup_{\ell \in L} C_\ell$$

is an upper bound for C in P. To show that J is an ideal, it suffices to check that J is closed under addition and scalar multiplication (on each side). It is clear that J is closed under scalar

[56] This is a critical step. Everybody that fouls up a Zorn's Lemma argument fouls it up by applying Zorn's Lemma to the empty set and thereby creating an element of the empty set with interesting properties!! All of which is absurd.

multiplication because each C_ℓ is. Use that fact that C is a chain to see that J is closed under addition. If x and x' are in J, then $x \in C_\ell$ and $x' \in C_{\ell'}$ for some ℓ and ℓ' in L. The set C is a chain; so $C_\ell \subseteq C_{\ell'}$ or $C_{\ell'} \subseteq C_\ell$. In either event, x and x' are both in one ideal from C; hence the sum is in this same ideal; therefore, the sum is in J. It is clear that J contains I and that $C_\ell \subseteq J$ for all $\ell \in L$. Finally, we show that J is a proper ideal. If not, then $1 \in J = \cup_\ell C_\ell$. Thus, $1 \in C_\ell$ for some ℓ; but each C_ℓ is a proper ideal so this did not happen.

We have established that every chain in P has an upper bound in P.

We apply Zorn's Lemma to conclude that P has a maximal element. It is clear that this maximal element of P is actually a maximal ideal of R which contains I. □

Theorem. 5.50 *If $\boldsymbol{k}$ is an arbitrary field, then there exists an algebraic closure K of $\boldsymbol{k}$.*

Proof. The proof has three steps.

(a) There exists a field $\boldsymbol{k}_1$ such that if $f \in \boldsymbol{k}[x]$ is a polynomial of positive degree, then f has a root in $\boldsymbol{k}_1$.
(b) There exists an algebraically closed field L, with $\boldsymbol{k} \subseteq L$.
(c) There exists an algebraic closure K of $\boldsymbol{k}$.

(a) Let R be the polynomial ring

$$R = \boldsymbol{k}[\{x_f | f \in \boldsymbol{k}[x] \text{ is a monic polynomial of positive degree}\}].$$

We have adjoined one variable to $\boldsymbol{k}$ for each monic polynomial of positive degree. So, R is a polynomial ring in a huge number of variables. Let I be the ideal

$$I = (\{f(x_f) \mid f \in \boldsymbol{k}[x] \text{ is a monic polynomial of positive degree}\}).$$

We will show that I is a proper ideal of R. Once that is accomplished, then we apply Theorem 5.56 to see that there exists a maximal ideal $\mathfrak{m}$ of R with $I \subseteq \mathfrak{m}$. Let $\boldsymbol{k}_1 = R/\mathfrak{m}$. Observe that $\boldsymbol{k}_1$ is a field. Observe also that if f is a monic polynomial of positive degree in $\boldsymbol{k}[x]$, then $\overline{x_f}$ is an element of $\boldsymbol{k}_1$ with $f(\overline{x_f}) = 0$ in $\boldsymbol{k}_1$.

We now show that I is a proper ideal of R. Suppose not. We will exhibit a contradiction. If I is not a proper ideal of R, then $1 \in I$. In other words, there are monic polynomials of positive degree $f_1, \dots, f_n$ in $\boldsymbol{k}[x]$ and there are elements $g_1, \dots, g_n \in R$ such that

$$1 = \sum_{i=1}^{n} g_i f_i(x_{f_i}) \tag{5.56.1}$$

in R. Let E be a splitting field for the product $f_1 f_2 \cdots f_n$ over $\boldsymbol{k}$. Observe that there exists $\alpha_1, \dots, \alpha_n$ in E such that $f_i(\alpha_i) = 0$ in E.

Let

$$\phi_1 : R = \boldsymbol{k}[\{x_f \mid f \in \boldsymbol{k}[x] \text{ is a monic polynomial of positive degree}\}] \to E[\{x_f \mid f \in \boldsymbol{k}[x] \text{ is a monic polynomial of positive degree}\}]$$

be the ring homomorphism which is induced by the inclusion of fields $\boldsymbol{k} \subseteq E$, and let

$$\phi_2 : E[\{x_f \mid f \in \boldsymbol{k}[x] \text{ is a monic polynomial of positive degree}\}] \to E[\{x_f \mid f \in \boldsymbol{k}[x] \text{ is a monic polynomial of positive degree}\}]$$

be the ring homomorphism which is the identity map on E, sends x_{f_i} to α_i for $1 \leq i \leq n$, and is the identity map on all of the other variables x_f. Let

$$\phi : R \to E[\{x_f \mid f \in \boldsymbol{k}[x] \text{ is a monic polynomial of positive degree}\}]$$

be the ring homomorphism which the the composition $\phi_2 \circ \phi_1$. Observe that ϕ carries (5.56.1) to

$$1 = \sum_{i=1}^{n} \phi(g_i) f_i(\alpha_i);$$

thus,

$$1 = 0$$

in the field E. This is a contradiction. We conclude that I is a proper ideal of R.

(b) Apply (a) infinitely many times to produce a countable infinite ascending chain of fields

$$\boldsymbol{k} = \boldsymbol{k}_0 \subseteq \boldsymbol{k}_1 \subseteq \boldsymbol{k}_2 \subseteq \dots$$

with the property that every polynomial in $\boldsymbol{k}_i[x]$ has a root in $\boldsymbol{k}_{i+1}$. Let L be $L = \bigcup_{i=1}^{\infty} \boldsymbol{k}_i$. Observe that L is a field. Observe, also, that every polynomial in $L[x]$ is also a polynomial in $\boldsymbol{k}_i[x]$, for some large i, and hence has a root in $\boldsymbol{k}_{i+1}$ (and therefore also in L). Thus L is algebraically closed.

(c) Let

$$K = \{\ell \in L | \ell \text{ is algebraic over } \boldsymbol{k}\}.$$

Observe that K is a field; K is algebraic over $\boldsymbol{k}$, and K is algebraically closed. Thus, K is an algebraic closure of $\boldsymbol{k}$. □

We move in the direction of proving the following result.

Theorem. 5.51 *If $\boldsymbol{k}$ is an arbitrary field, and K_1 and K_2 are algebraic closures of $\boldsymbol{k}$, then K_1 and K_2 are isomorphic over*[55] *$\boldsymbol{k}$.*

We begin with a (fairly obvious) Lemma. (The proof we are about to give is essentially the same as the proof of (c) implies (a) in Theorem 5.30 on page 108.)

Lemma 5.57. *Let $\boldsymbol{k} \subseteq E_1 \subseteq E$ be fields. Suppose $\boldsymbol{k} \subseteq K$ is another field extension, with K algebraically closed, and $\sigma : E_1 \to K$ is a field homomorphism over $\boldsymbol{k}$. Let e be an element of E which is algebraic over E_1. Then there is an field homomorphism $\widetilde{\sigma} : E_1[e] \to K$ which extends σ.*

Proof. Let $f = \sum_i \alpha_i x^i$ be the minimal polynomial of e over E_1. Let $f^\sigma = \sum_i \sigma(\alpha_i) x^i$ be the image of f in $(\sigma E_1)[x]$. Observe that f^σ is an irreducible polynomial in $(\sigma E)[x]$. The field K is algebraically closed; so there is a root u of f^σ in K. The First Isomorphism Theorem gives isomorphisms

$$\phi_1 : \frac{E_1[x]}{(f)} \to E_1[e] \quad \text{and} \quad \phi_2 : \frac{(\sigma E_1)[x]}{(f^\sigma)} \to (\sigma E_1)[u],$$

with $\phi_1(\bar{x}) = e$ and $\phi_2(\bar{x}) = u$. Furthermore σ induces an isomorphism $\phi_3 : \frac{E_1[x]}{(f)} \to \frac{(\sigma E_1)[x]}{(f^\sigma)}$. Observe that $\widetilde{\sigma} = \phi_2 \circ \phi_3 \circ \phi_1^{-1} : E_1[e] \to (\sigma E_1)[u]$ is an isomorphism of fields which extends σ. □

Proof of Theorem 5.51. Let

$$P = \{(E, \sigma) \mid \boldsymbol{k} \subseteq E \subseteq K_1 \text{ are fields and } \sigma : E \to K_2 \text{ is a homomorphism of fields over } \boldsymbol{k}\}.$$

If (E, σ) and (E', σ') are elements of E, then write

$$(E, \sigma) \le (E', \sigma')$$

if $E \subseteq E'$ and $\sigma'|_E = \sigma$. Observe that $(P, \le)$ is a partially ordered set. The set P is non-empty because $(\boldsymbol{k}, \mathrm{id})$ is in P. Suppose that

$$C = \{(E_\ell, \sigma_\ell) \mid \ell \in L\}$$

is a chain in P. Let E be the set $\cup_{\ell\in L} E_\ell$, and define $\sigma : E \to K_2$ by $\sigma(e) = \sigma_\ell(e)$ for all ℓ with $e \in E_\ell$. The set E is a field because C is a chain; similarly, σ is a well-defined field homomorphism because C is a chain. It follows that (E, σ) is an upper bound for C in P. Apply Zorn's Lemma to obtain a maximal element $(E_{\max}, \sigma_{\max})$ in S.

Observe that $E_{\max} = K_1$. Otherwise, we apply Lemma 5.57 to produce an element $(E_{\max}[e], \widetilde{\sigma})$ in P with $(E_{\max}, \sigma_{\max})$ strictly less than $(E_{\max}[e], \widetilde{\sigma})$.

Observe that $\boldsymbol{k} \subseteq \sigma(K_1) \subseteq K_2$with $\sigma(K_1)$ an algebraically closed field and K_2 algebraic over $\boldsymbol{k}$. Thus, every element of K_2 is algebraic over the algebraically closed field $\sigma(K_1)$. It follows that $\sigma(K_1) = K_2$. □

5.G. **Solvable groups.** [57]

Definition 5.58. The group G is called solvable if there exist subgroups

$$\langle \mathrm{id} \rangle = G_n \lhd G_{n-1} \lhd G_{n-2} \lhd \cdots \lhd G_1 \lhd G_0 = G$$

with $\frac{G_i}{G_{i+1}}$ an Abelian group for all i.

Examples 5.59.

(a) Every Abelian group is solvable.

(b) If G is a simple, non-Abelian, group, then G is not solvable. (In particular, we will prove that A_n is a simple group for $5 \le n$. As a consequence, A_n is not a solvable group for $5 \le n$. We will also see that if G is a solvable group, then every subgroup of G and every homomorphic image of G is solvable. In particular, it will follow that S_n is not a solvable group for $5 \le n$.

(c) S_3 and S_4 are solvable groups; because:

$$1 \lhd A_3 \lhd S_3 \quad \text{and} \quad 1 \lhd V_4 \lhd A_4 \lhd S_4$$

and the appropriate quotient groups are Abelian. See example 2.47, if necessary.

[57] In Theorem 5.69 we will prove that if $f \in \boldsymbol{k}[x]$, then f is solvable by radical (that is, the solutions of $f(x) = 0$ in some splitting field of f over $\boldsymbol{k}$ may expressed in terms of field operations and the taking of $\sqrt[n]{}$ for various n, starting, of course, with elements of $\boldsymbol{k}$) if and only if the group $\mathrm{Aut}_{\boldsymbol{k}} K$ is a solvable group, where K is the splitting field of f. In the present section we learn that the symmetric group S_n is not a solvable group for $n \le 5$. In sections 5.I and 5.L, we will also learn that some polynomials over $\boldsymbol{k} = \mathbb{Q}$ have $\mathrm{Aut}_{\boldsymbol{k}} K = S_n$, where again K is the splitting field of f over $\boldsymbol{k}$. The ultimate point is that there is no procedure for solving $f(x) = 0$ where f has rational coefficients and degree five or more in terms of the field operations and the taking of $\sqrt[n]{}$ for various n.

(d) Every p-group is solvable. Indeed, every p-group has a non-trivial center. (See 2.72, if necessary.) Let G be a p-group. Let C_1 be the center of G. Let C_2 be the subgroup of G with $\frac{C_2}{C_1}$ equal to the center of $\frac{G}{C_1}$. Continue in this manner to produce a sequence of subgroups $\{C_i\}$ of G with $\frac{C_i}{C_{i-1}}$ equal to the center of $\frac{G}{C_{i-1}}$,

We prove two results in this section

Theorem 5.60. *If $5 \le n$, then A_n is a simple group.*

Theorem 5.61. *If G is a solvable group, then every subgroup of G and every homomorphic image of G is also solvable.*

We move in direction of proving Theorem 5.60.

Lemma 5.62. *Let N be a normal subgroup of A_n. If N contains a three cycle, then $N = A_n$.*

Proof. There is nothing to prove if $n \le 2$. If $n = 3$, then each of the 3-cycles is the generator for the cyclic group A_3. It suffices to prove the assertion for $4 \le n$. It is a two step process.

Step 1. We prove that if N contains at least one 3-cycle, then N contains every 3-cycle.
Step 2. We prove that if N contains every 3-cycle, then N is A_n.

Proof of Step 1. One may iterate. Thus, it suffices to show that

$$(abc) \in N \text{ and } d \notin \{a, b, c\} \Rightarrow (abd) \in N.$$

This is very easy. Observe that $(abc) \in N$ implies[58]

$$(adc)(abc)(adc)^{-1} = (adb) \in N.$$

Proof of Step 2. Observe that if a, b, c, d are distinct, then

$$(abd)(abc) = (ad)(bc)$$

and if a, b, c are distinct, then

$$(abc) = (ac)(ab).$$

□

The proof of Theorem 5.60. In light of Lemma 5.62, it suffices to prove that if $5 \le n$ and N is a normal subgroup of A_n, then N contains a 3-cycle.

Notice first that if N contains an element of the form $(ab)(cd)$ with a, b, c, d distinct elements in $\{1, \dots, n\}$, then N contains a three cycle. Indeed, let e be an element in $\{1, \dots, n\}$ with a, b, c, d, e distinct, then

$$(ab)(de) = (cde)\Big((ab)(cd)\Big)(cde)^{-1} \in N \qquad (cde) = \Big((ab)(cd)\Big)\Big((ab)(de)\Big) \in N$$

[58] Of course, I am using Observation 2.46.

Let α be a non-identity element of N with the maximum number of fixed points. We claim that α is a 3-cycle. Otherwise, α looks like one of the following:[59]

$$\begin{gathered}(abc)(de\dots\\(abcf)(de\dots\\(abcde\dots\\(ab)(cd)(ef)\dots\end{gathered}$$

for distinct elements a, b, c, d, e, (and, when needed, f) in $\{1,\dots,n\}$. Let $\beta=(cde)\alpha(cde)^{-1}$. Observe that β and $\beta\circ\alpha^{-1}$ are elements of N. Observe also that β is equal to

$$\begin{gathered}(abd)(ec\dots\\(abdf)(ec\dots\\(abdec\dots\\(ab)(de)(cf)\dots\end{gathered}$$

It is clear that $\beta\neq\alpha$ (indeed, α of c and β of c are different in all cases) and α and β both move a, b, c, d, and e. The permutation α carries a to b; so $\alpha^{-1}(b)=a$ and $\beta\circ\alpha^{-1}$ leaves b fixed. Furthermore, every integer that is left fixed by α is also left fixed by $\beta\circ\alpha^{-1}$. Thus, $\beta\circ\alpha^{-1}$ is a non-identity element of N which fixes more integers than than α fixes. This contradicts the choice of α. Hence, N does contain a 3-cycle and the proof is complete. □

We head in the direction of proving

Theorem. 5.61 *If G is a solvable group, then every subgroup of G and every homomorphic image of G is also solvable.*

The idea is that the original definition of solvable said that the group G is solvable if there exists subgroups $G_1\supseteq G_2\supseteq\cdots$ with some property. If one uses the original definition to decide if the subgroup H of G or the homomorphic image $\frac{G}{N}$ of the solvable group G is solvable, then it appears that one has to start from scratch and look for a chain of subgroups of H or of $\frac{G}{N}$. Instead we would like to have an algorithm for determining if a group is solvable. If the algorithm says that G is solvable, then there is a chance that we can directly deduce the outcome of the algorithm when it is applied to H or $\frac{G}{N}$. Roughly speaking, at each step our algorithm identifies the smallest normal subgroup of what ever group we are considering such that the quotient group of the group under consideration by this subgroup is Abelian.

Definition 5.63. If G is a group, then the commutator subgroup of G is the subgroup of G which is generated by

$$\{hgh^{-1}g^{-1}\mid h,g\in G\}.$$

The commutator subgroup of G is denoted by G' or $G^{(1)}$ and the quotient $\frac{G}{G'}$ is called the Abelianization of G. For each integer i, with $2\le i$, define $G^{(i)}=G^{(i-1)\prime}$.

[59] The point is that if α has a factor of a 3-cycle, then at least two more integers have to be moved. If α has a factor of a four cycle, then α can not be a four cycle, because a four cycle is an odd permutation; so once again two more integers have to be moved. If α is a product of disjoint transpositions, then α must be a product of at least 4 such transpositions because we already saw that α is not the product of two disjoint transpositions and the number of transpositions must be even.

Proposition 5.64. *If G is a group, then the following statements hold*:

(a) $G' \lhd G$,

(b) $\frac{G}{G'}$, *is an Abelian group, and*

(c) *if $N \lhd G$ and $\frac{G}{N}$ is Abelian, then $G' \lhd N$.*

Remark 5.65. One can phrase (c) as a Universal Mapping Property. Let G be a group. The Abelianization of G is an Abelian group A together with a group homomorphism $\pi : G \to A$ with the property that whenever B is an Abelian group and $\phi : G \to B$ is a group homomorphism, then there exists a unique group homomorphism $\widetilde{\phi} : A \to B$ such that $\widetilde{\phi} \circ \pi = \phi$. This information is captured by stating that the diagram

$$\begin{tikzcd}
G \arrow[r, "\phi"] \arrow[d, "\pi"'] & B \\
A \arrow[ur, dashed, "\exists!\widetilde{\phi}"'] &
\end{tikzcd}$$

commutes.

The proof of Proposition 5.64. The Proposition is essentially obvious. The commutator subgroup G' is defined to a subgroup of G. To show that G' is a normal subgroup, it suffices to show that $a[h,g]a^{-1}$ is in G' for $a, g, h \in G$, where

$$[h,g] = hgh^{-1}g^{-1}$$

is the "commutator of h and g". This is clear because

$$a[h,g]a^{-1} = [aha^{-1}, aga^{-1}].$$

It is clear that G/G' is an Abelian group. If $\phi : G \to B$ is a group homomorphism and B is an Abelian group, then each commutator $[h,g]$ is in the kernel of ϕ. Apply the first isomorphism theorem. □

Lemma 5.66. *The group G is solvable if and only is $G^{(s)} = \langle \mathrm{id} \rangle$ for some s.*

Proof.
($\Leftarrow$) If $G^{(s)} = \langle \mathrm{id} \rangle$, then

$$\langle \mathrm{id} \rangle = G^{(s)} \lhd G^{(s-1)} \lhd \cdots \lhd G^{(1)} \lhd G^{(0)} = G$$

with $\frac{G^{(i)}}{G^{(i+1)}}$ Abelian for $0 \le i \le s-1$. Thus, G is a solvable group.

($\Rightarrow$) Suppose G is a solvable group. Then there are subgroups

$$\langle \mathrm{id} \rangle = G_n \lhd G_{n-1} \lhd G_{n-2} \lhd \cdots \lhd G_1 \lhd G_0 = G$$

with each $\frac{G_i}{G_{i+1}}$ an Abelian group.

We claim that $G^{(i)} \subseteq G_i$ for each i. Apply Proposition 5.64.(c) to the Abelian group $\frac{G}{G_1}$ to see that $G^{(1)} \subseteq G_1$. Assume by induction that $G^{(i)} \subseteq G_i$. Take the commutator subgroup of each side to obtain

$$G^{(i+1)} = G^{(i)\prime} \subseteq G_i' \tag{5.66.1}$$

The fact that $\frac{G_i}{G_{i+1}}$ is Abelian implies (again by way of Proposition 5.64.(c)) that

$$G_i' \subseteq G_{i+1}. \tag{5.66.2}$$

Combine (5.66.1) and (5.66.2) to see that

$$G^{(i+1)} \subseteq G_i' \subseteq G_{i+1},$$

and the claim is established.

In particular, $G^{(n)} \subseteq G_n = \langle \mathrm{id} \rangle$. □

We now prove

Theorem. 5.61 *If G is a solvable group, then every subgroup of G and every homomorphic image of G is also solvable.*

Proof. If H is a subgroup of the solvable group G, then $H^{(i)}$ is a subgroup of $G^{(i)}$ for all i. If $G^{(s)} = \langle \mathrm{id} \rangle$, then $H^{(s)} = \langle \mathrm{id} \rangle$. Apply Lemma 5.66. If $\phi : G \to \bar{G}$ is a surjective group homomorphism, then $\phi(G^{(i)}) = \bar{G}^{(i)}$ for all i. If $G^{(s)} = \langle \mathrm{id} \rangle$, then $\bar{G}^{(s)} = \langle \mathrm{id} \rangle$. □

Corollary 5.67. *If N is a normal subgroup of a group G, with N solvable and G/N solvable, then G is solvable.*

Proof. Write G/N as $\bar{G}$ and let $\phi : G \to \bar{G}$ be the natural quotient map. Observe that

$$\phi(G^{(i)}) = \bar{G}^{(i)}$$

for all i. The hypothesis that $\bar{G}$ is solvable guarantees that $\bar{G}^{(s)} = \langle \mathrm{id} \rangle$ for some s. It follows that $G^{(s)} \subseteq N$. On the other hand N is also solvable; hence $N^{(t)} = \langle \mathrm{id} \rangle$ for some t. Thus, $G^{(s+t)} = \langle \mathrm{id} \rangle$ and G is solvable. □

5.H. **Solvability by radical.**

In this section, the field $\boldsymbol{k}$ has characteristic zero.[60]

Definition 5.68. Let $\boldsymbol{k}$ be a field of characteristic zero. The polynomial f in $\boldsymbol{k}[x]$ is solvable by radical if there exist fields

$$\boldsymbol{k} = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_r$$

with the splitting field of f contained in K_r and $K_i = K_{i-1}[d_i]$, where $d_i^{n_i} \in F_{i-1}$ for some n_i.

The purpose of this section is to prove Theorem 5.69.

Theorem 5.69. *Let $\boldsymbol{k}$ be a field of characteristic zero. The polynomial $f \in \boldsymbol{k}[x]$ is solvable by radical if and only if the group $\mathrm{Aut}_{\boldsymbol{k}} K$ is solvable, where K is the splitting field of f over $\boldsymbol{k}$.*

[60] Actually, the fields in Lemmas 5.73 and 5.76 do not have to have characteristic zero as long as the stated hypotheses are satisfied. However these hypotheses automatically put restrictions on the characteristic.

We first prove the direction "⇒"; that is, we assume f is solvable by radical and we prove that $\operatorname{Aut}_{\boldsymbol{k}} K$ is a solvable group. In order to do this we must figure out the Galois group for two types of extensions. In 5.72 we figure out the Galois group when roots of 1 are adjoined to a field $\boldsymbol{k}$. In 5.73, we assume that the n^{th} roots of 1 are already in $\boldsymbol{k}$ and we figure out the Galois group that corresponds to adjoining an n^{th} root of α to $\boldsymbol{k}$ for some $\alpha \in \boldsymbol{k}$. Observation 5.70 (together with the remarks of 5.71) is a little doodle that is the key to the proof of 5.72.

Observation 5.70. *If G is a finite cyclic group of order n, then the group* $\operatorname{Aut} G$ *is isomorphic to* $\left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^*$.

The observation is essentially obvious. I will merely spell out slowly what is being asserted.

Remarks 5.71.

- A group isomorphism from G to G is called an automorphism of G. The set of automorphisms of G is denoted by $\operatorname{Aut} G$. This set forms a group under composition. (We surely talked about this in 701.)
- Recall that $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is a ring. Some of the elements of this ring are multiplicative units; that is, they have multiplicative inverses in $\frac{\mathbb{Z}}{n\mathbb{Z}}$. In the statement of 5.70, $\left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^*$ represents the set of multiplicative units in $\frac{\mathbb{Z}}{n\mathbb{Z}}$; this set of multiplicative units forms a group under multiplication.
- If r is an integer, then let $\bar{r}$ represent the class of r in $\frac{\mathbb{Z}}{n\mathbb{Z}}$. Observe that
$$\left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^* = \left\{ \bar{r} \in \frac{\mathbb{Z}}{n\mathbb{Z}} \;\middle|\; r \text{ and } n \text{ are relatively prime as integers.} \right\}$$
- Observe that for each integer r, the function $\sigma_r : G \to G$ with $\sigma_r(g) = g^r$, for all $g \in G$, is a well-defined group homomorphism. Furthermore, σ_r is a group automorphism if and only if r and n relatively prime.
- Observe that the function
$$\left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^* \to \operatorname{Aut} G$$
which sends $\bar{r}$ to σ_r is a well-defined isomorphism of Abelian groups.

Lemma 5.72. *Let $\boldsymbol{k}$ be a field of characteristic zero and let K be the splitting field of $f = x^n - 1$. The following statements hold.*

(a) *The field extension $\boldsymbol{k} \subseteq K$ is Galois.*
(b) *The roots of $x^n - 1$ form a cyclic subgroup of the group $(K \setminus \{0\}, \times)$ of order n.*
(c) *The group* $\operatorname{Aut}_{\boldsymbol{k}} K$ *is Abelian.*

Proof. Assertion (a) is obvious! The field $\boldsymbol{k}$ has characteristic zero; hence every polynomial in $\boldsymbol{k}[x]$ is separable. The field K is the splitting field of a separable over $\boldsymbol{k}$; therefore, the field extension $\boldsymbol{k} \subseteq K$ is Galois.

If a polynomial $g \in \boldsymbol{k}[x]$ has a repeated root (in the splitting field of g over $\boldsymbol{k}$), then g and g' have an irreducible factor in common in $\boldsymbol{k}[x]$. The polynomials $f = x^n - 1$ and $f' = nx^{n-1}$ do not have any factors in common in $\boldsymbol{k}[x]$; so f has n distinct[61]roots in K.

It is clear that the roots of $x^n - 1$ form a subgroup of $(K \setminus \{0\}, \times)$ under multiplication. See Homework problem 12(c) for a proof that every finite subgroup of $(K \setminus \{0\}, \times)$ is cyclic. This completes the proof of (b).

Now we prove (c). Let U be the group of roots of $x^n - 1$ in K. Apply Observation 5.70 and (b) to see that $\operatorname{Aut} U$ is isomorphic to the Abelian group $\left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^*$, which is an Abelian group. We complete the proof by observing that the function

$$\operatorname{Aut}_{\boldsymbol{k}} K \longrightarrow \operatorname{Aut} U,$$

which sends σ to $\sigma|_U$ is an injective group homomorphism.

The assertion of the preceding sentence is fairly obvious; but it also is very clever. So be sure to take the time necessary to understand it. If σ is an automorphism of K over $\boldsymbol{k}$, then σ carries each root of $x^n - 1$ to a root of $x^n - 1$. Thus, σ restricts to become a set bijection[62] from the set of roots of $x^n - 1$ to itself. On the other hand, the roots of $x^n - 1$ actually form a group (which we called U); and the restriction of σ to U is a group homomorphism! Thus $\sigma|_U$ is in $\operatorname{Aut} U$. Furthermore, K is the splitting field of $x^n - 1$ over $\boldsymbol{k}$; so K is equal to $\boldsymbol{k}$ with the roots of $x^n - 1$ adjoined. It follows that an element σ of $\operatorname{Aut}_{\boldsymbol{k}} K$ is completely determined by the action of $\sigma|_U$; thus, the function from $\operatorname{Aut}_{\boldsymbol{k}} K$ to $\operatorname{Aut} U$, which is given by $\sigma \mapsto \sigma|_U$, is injective. □

Lemma 5.73. *Let $\boldsymbol{k} \subseteq K$ be fields.*[60] *Assume that $\boldsymbol{k}$ contains n distinct n^{th} roots of* 1 *and $K = \boldsymbol{k}[d]$ where $d^n \in \boldsymbol{k}$. Then $\boldsymbol{k} \subseteq K$ is a Galois extension and the group* $\operatorname{Aut}_{\boldsymbol{k}} K$ *is a cyclic group.*

Proof. Let U be the set of n^{th} roots of 1 in K. It is clear that U forms a finite subgroup of $(K \setminus \{0\}, \times)$. Recall, once again, that every finite subgroup of $(K \setminus \{0\}, \times)$ is cyclic. Identify $\omega \in K$ so that

$$U = \{\omega^i | 0 \le i \le n-1\}.$$

In $K[x]$, the polynomial $x^n - d^n$ is equal to $\prod_{i=0}^{n-1}(x - \omega^i d)$. In particular, K is the splitting field of a separable polynomial over $\boldsymbol{k}$; and therefore, the field extension $\boldsymbol{k} \subseteq K$ is Galois. If $\sigma \in \operatorname{Aut}_{\boldsymbol{k}} K$, then $\sigma(d) = \omega^i d$ for some i and $\frac{\sigma(d)}{d} = \omega^i$; thus,

$$\operatorname{Aut}_{\boldsymbol{k}} K \to U,$$

[61] Please notice that assertion (b) is NOT completely obvious and it most certainly does NOT follow from assertion (a). Indeed, the notion of separable is sneaky. Recall from Definition 5.33 that the polynomial $f \in \boldsymbol{k}[x]$ is separable if every irreducible factor of f in $\boldsymbol{k}[x]$ factors into distinct linear factors in some normal extension K of $\boldsymbol{k}$. Here is the point. If $\boldsymbol{k}$ is a field of characteristic p, then the polynomial $x^p - 1$ is a separable polynomial since $x^p - 1 = (x-1)^p$ and the only irreducible factor of $x^p - 1$ is $x - 1$, which does not have repeated roots anywhere! If K is the splitting field of $x^p - 1$ over $\boldsymbol{k}$, then the field extension $\boldsymbol{k} \subseteq K$ is Galois (since $K = \boldsymbol{k}$) but of course, K does NOT contain p distinct p^{th}-roots of 1. The only p^{th} root of 1 in K is 1.

[62] This is a good time to remember that when Galois developed Galois Theory, he did not have any of the language that we now use. The notions of "groups" and "fields" came fifty years later. Given a polynomial (probably with rational coefficients) he considered a set of distinguished permutations of its roots (probably in $\mathbb{C}$). Each of Galois' distinguished permutations is the restriction to the roots of f of one of our field automorphisms.

given by $\sigma \mapsto \frac{\sigma(d)}{d}$ is a well-defined function. Observe that this function is also an injective group homomorphism. We have shown that $\mathrm{Aut}_{\boldsymbol{k}} K$ is isomorphic to a subgroup of a cyclic group. It follows that $\mathrm{Aut}_{\boldsymbol{k}} K$ is a cyclic group. □

The proof of the direction (⇒) of Theorem 5.69.

We are given a field $\boldsymbol{k}$ of characteristic zero, a polynomial $f \in \boldsymbol{k}[x]$, the splitting field K of f over $\boldsymbol{k}$, and a chain of fields

$$\boldsymbol{k} \subseteq \boldsymbol{k}[d_1] \subseteq \boldsymbol{k}[d_1, d_2] \subseteq \cdots \subseteq \boldsymbol{k}[d_1, \dots d_r] \tag{5.73.1}$$

with $K \subseteq \boldsymbol{k}[d_1, \dots d_r]$ and $d_i^{n_i} \in \boldsymbol{k}[d_1, \dots, d_{i-1}]$, for some n_i, for each i.

We want to prove that $\mathrm{Aut}_{\boldsymbol{k}} K$ is a solvable group.

The first thing we do is we replace the tower of fields (5.73.1) with a "better" tower of fields. The better tower of fields will have two more properties:

(a) We will adjoin all relevant roots of 1 right at the beginning, so that we can appeal to Lemma 5.73 at all of the rest of the extensions.

(b) The extension from $\boldsymbol{k}$ to the top field in the better tower will be a Galois extension so that we can use the Fundamental Theorem of Galois Theory.

Let n be the least[63] common multiple of the n_i and let d_0 be a generator for the group n distinct n^{th} roots of 1 in some extension field of $\boldsymbol{k}[d_1, \dots d_r]$. Let g_i be the minimal polynomial of d_i over $\boldsymbol{k}$, for $0 \le i \le r$ and let E be the splitting field of $\prod_{i=0}^{r} g_i$ over $\boldsymbol{k}$. The characteristic of $\boldsymbol{k}$ is zero, so every polynomial in $\boldsymbol{k}[x]$ is separable. The field E is the splitting field of a separable polynomial over $\boldsymbol{k}$; thus, the field extension $\boldsymbol{k} \subseteq E$ is Galois.

We now create a tower of fields between $\boldsymbol{k}$ and E which has the property of the tower of fields in (5.73.1). Let $\mathrm{Aut}_{\boldsymbol{k}} E = \{\sigma_1 = \mathrm{id}, \sigma_2, \dots, \sigma_N\}$, for some number N. Observe that

$$E = \boldsymbol{k}[d_0, d_1, \dots, d_r, \sigma_2(d_1), \dots, \sigma_2(d_r), \sigma_3(d_1), \dots, \sigma_3(d_r), \dots, \sigma_N(d_1), \dots, \sigma_N(d_r)].$$

We plan to create a tower of fields by adding one element at a time from the listed generators of E. We start with $\boldsymbol{k}$. We adjoin the elements in the order they are listed:

$$\boldsymbol{k} \subseteq \boldsymbol{k}[d_0] \subseteq \boldsymbol{k}[d_0, d_1] \subseteq \cdots \subseteq \boldsymbol{k}[d_0, d_1, \dots, d_r] \subseteq \boldsymbol{k}[d_0, d_1, \dots, d_r, \sigma_2(d_1)] \subseteq \cdots \subseteq E.$$

There is no need to adjoin $\sigma_i(d_0)$ for $2 \le i$ because $\sigma_i(d_0)$ is in the multiplicative group generated by d_0. Observe that

- $d_0^n \in \boldsymbol{k}$,
- $d_i^{n_i} \in \boldsymbol{k}[d_0, d_1, \dots, d_{i-1}]$, for $1 \le i \le r$,
- $(\sigma_2 d_1)^{n_1} = \sigma_2(d_1^{n_1}) \in \boldsymbol{k}[d_0, d_1, \dots, d_r]$, because $d_1^{n_1} \in \boldsymbol{k}$ and σ_2 leaves $\boldsymbol{k}$ fixed,
- $(\sigma_2 d_2)^{n_2} = \sigma_2(d_2^{n_2}) \in \boldsymbol{k}[d_0, d_1, \dots, d_r, \sigma_2 d_1]$, because $d_2^{n_2} \in \boldsymbol{k}[d_1]$ and σ_2 leaves $\boldsymbol{k}$ fixed,
- etc.

[63] Actually, any common multiple would do.

There is no need to preserve the names we are presently using; consequently, we switch to more convenient names. At this point we have a chain of fields

$$\boldsymbol{k} = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m = E$$

that satisfies the following properties.

- The field extension $\boldsymbol{k} \subseteq E$ is Galois extension.
- The field extension $E_0 \subseteq E_1$ is the Galois extension which is obtained by adjoining n distinct n^{th} roots of 1 to E_0. In particular, we know from Lemma 5.72, that the group

$$\operatorname{Aut}_{E_0} E_1 \text{ is Abelian.} \tag{5.73.2}$$

- If $2 \le i \le m$, them $E_i = E_{i-1}[\delta_i]$ for some δ_i with $\delta_i^{n_i} \in E_{i-1}$, for some n_i, and E_{i-1} contains n_i distinct n_i^{th}-roots of 1. In particular, we know from Lemma 5.73 that $E_{i-1} \subseteq E_i$ is a Galois extension and the Galois group

$$\operatorname{Aut}_{E_{i-1}} E_i \text{ is a cyclic group.} \tag{5.73.3}$$

We prove that $\operatorname{Aut}_{\boldsymbol{k}} E$ is a solvable group.

Please notice that the original field of interest, K, satisfies $\boldsymbol{k} \subseteq K \subseteq E$ with $\boldsymbol{k} \subseteq K$ and $\boldsymbol{k} \subseteq E$ both Galois extensions. It follows from Theorem 5.20.(d) that

$$\operatorname{Aut}_K E \lhd \operatorname{Aut}_{\boldsymbol{k}} E \quad \text{and} \quad \operatorname{Aut}_{\boldsymbol{k}} K \cong \frac{\operatorname{Aut}_{\boldsymbol{k}} E}{\operatorname{Aut}_K E};$$

thus, $\operatorname{Aut}_{\boldsymbol{k}} K$ is a homomorphic image of $\operatorname{Aut}_{\boldsymbol{k}} E$. Once we prove that $\operatorname{Aut}_{\boldsymbol{k}} E$ is a solvable group, then it follows from Theorem 5.61, that $\operatorname{Aut}_{\boldsymbol{k}} K$ is also solvable group.

We prove that $\operatorname{Aut}_{\boldsymbol{k}} E$ is a solvable group by showing that

$$\langle \mathrm{id} \rangle = \operatorname{Aut}_{E_m} E \lhd \operatorname{Aut}_{E_{m-1}} E \lhd \cdots \lhd \operatorname{Aut}_{E_0} E = \operatorname{Aut}_{\boldsymbol{k}} E,$$

with

$$\frac{\operatorname{Aut}_{E_i} E}{\operatorname{Aut}_{E_{i+1}} E}$$

Abelian for all i with $0 \le i \le m-1$.

On the other hand, for each i, the fields

$$E_i \subseteq E_{i+1} \subseteq E$$

satisfy $E_i \subseteq E_{i+1}$ is a Galois extension and $E_i \subseteq E$ is a Galois extension.[64] So we apply Theorem 5.20.(d) to conclude that

$$\operatorname{Aut}_{E_{i+1}} E \lhd \operatorname{Aut}_{E_i} E$$

and that

$$\operatorname{Aut}_{E_i} E_{i+1} \cong \frac{\operatorname{Aut}_{E_i} E}{\operatorname{Aut}_{E_{i+1}} E}.$$

Of course, we know from (5.73.2) and (5.73.3) that $\operatorname{Aut}_{E_i} E_{i+1}$ is an Abelian group. □

[64] We used Lemmas 5.72 and 5.73 to learn that $E_i \subseteq E_{i+1}$ is a Galois extension and we applied Theorem 5.20.(c) to the chain of fields $\boldsymbol{k} \subseteq E_i \subseteq E$ to learn that $E_i \subseteq E$ is a Galois extension.

April 15, 2024

Observation 5.74. *Let $\boldsymbol{k} \subseteq K$ be a finite dimensional Galois extension. Then* $\operatorname{Aut}_{\boldsymbol{k}} K$ *is a solvable group if and only if there are fields*

$$\boldsymbol{k} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{n-1} \subseteq K_n = K$$

such that $K_i \subseteq K_{i+1}$ is a Galois extension with an Abelian Galois group, for all i.

Proof. ($\Rightarrow$) Suppose $\operatorname{Aut}_{\boldsymbol{k}} K$ is a solvable group. Then, there exist subgroups

$$\langle \mathrm{id} \rangle = H_n \lhd H_{n-1} \lhd H_{n-2} \lhd \ldots \lhd H_1 \lhd H_0 = \operatorname{Aut}_{\boldsymbol{k}} K$$

with

$$\frac{H_i}{H_{i+1}}$$

an Abelian group for all i.

The corresponding fields are

$$\boldsymbol{k} = \underbrace{K^{H_0}}_{K_0} \subseteq \underbrace{K^{H_1}}_{K_1} \subseteq \cdots \subseteq \underbrace{K^{H_{n-1}}}_{K_{n-1}} \subseteq \underbrace{K^{H_n}}_{K_n} = K$$

Consider the fields

$$K_i \subseteq K_{i+1} \subseteq K.$$

Both fields K_i and K_{i+1} are intermediate between $\boldsymbol{k} \subseteq K$ (and this is a Galois extension) hence, $K_i \subseteq K$ and $K_{i+1} \subseteq K$ are both Galois extensions. The Galois group for $K_i \subseteq K$ is $\operatorname{Aut}_{K_i} K = H_i$. In a similar manner, the Galois group for $K_{i+1} \subseteq K$ is $\operatorname{Aut}_{K_{i+1}} K = H_{i+1}$. The subgroup $H_{i+1} \lhd H_i$; hence $K_i \subseteq K_{i+1}$ is a Galois extension and

$$\operatorname{Aut}_{K_i} K_{i+1} \cong \frac{\operatorname{Aut}_{K_i} K}{\operatorname{Aut}_{K_{i+1}} K} = \frac{H_i}{H_{i+1}},$$

and this is an Abelian group.

($\Leftarrow$) Consider fields

$$\boldsymbol{k} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{n-1} \subseteq K_n = K$$

such that $K_i \subseteq K_{i+1}$ is a Galois extension with an Abelian Galois group, for all i. The extension $\boldsymbol{k} \subseteq K$ ia Galois by hypothesis. We prove that $\operatorname{Aut}_{\boldsymbol{k}} K$ is solvable. Let $H_i = \operatorname{Aut}_{K_i} K$ for all i. Observe that

$$\langle \mathrm{id} \rangle = \underbrace{H_n}_{\operatorname{Aut}_{K_n} K} \subseteq \underbrace{H_{n-1}}_{\operatorname{Aut}_{K_{n-1}} K} \subseteq \underbrace{H_{n-2}}_{\operatorname{Aut}_{K_{n-2}} K} \subseteq \ldots \underbrace{H_1}_{\operatorname{Aut}_{K_{n-1}} K} \subseteq \underbrace{H_0}_{\operatorname{Aut}_{K_0} K} = \operatorname{Aut}_{\boldsymbol{k}} K.$$

The extension $K_i \subseteq K_{i+1}$ is Galois with an Abelian Galois group. Thus

$$\underbrace{\operatorname{Aut}_{K_{i+1}} K}_{H_{i+1}} \lhd \underbrace{\operatorname{Aut}_{K_i} K}_{H_i}$$

and

$$\frac{H_i}{H_{i+1}} = \frac{\operatorname{Aut}_{K_i} K}{\operatorname{Aut}_{K_{i+1}} K} \cong \operatorname{Aut}_{K_i} K_{i+1}$$

is an Abelian group. □

Theorem. 5.69 *Let $\boldsymbol{k}$ be a field of characteristic zero. The polynomial $f \in \boldsymbol{k}[x]$ is solvable by radical if and only if the group* $\operatorname{Aut}_{\boldsymbol{k}} K$ *is solvable, where K is the splitting field of f over $\boldsymbol{k}$.*

Summary of the proof of the direction (⇒) of Theorem 5.69. If we exhibit a finite dimensional extension E of K with $\boldsymbol{k} \subset E$ Galois with solvable Galois group, then we are done.

We did exhibit a finite dimensional extension E of K, with $\boldsymbol{k} \subseteq E$ Galois, and a collection of fields

$$\boldsymbol{k} = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{n-1} \subseteq E_n = E$$

with $E_i \subseteq E_{i+1}$ Galois with solvable Galois group. So, we are done with this direction.

Now we turn our attention to proving the (⇐) direction of Theorem 5.69. We start with a field $\boldsymbol{k}$ of characteristic zero, a polynomial $f \in \boldsymbol{k}[x]$, and a splitting field K of f over $\boldsymbol{k}$. We assume that $\operatorname{Aut}_{\boldsymbol{k}} K$ is a solvable group. We prove that f is solvable by radical. We start with two Lemmas. We use Lemma 5.75 to adjoin all relevant roots of 1 right away at the beginning. All of the heavy lifting is done in in Lemma 5.76.

Lemma 5.75. *Let*

$$\begin{tikzcd}
 & L \\
K \arrow[ur, hook] & \\
 & E \arrow[uu, hook] \\
\boldsymbol{k} \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}$$

be fields. Suppose that f is be a separable polynomial in $\boldsymbol{k}[x]$, K is the splitting field of f over $\boldsymbol{k}$ and L is the splitting field of f over E. Then $\operatorname{Aut}_E L$ *is isomorphic to a subgroup of* $\operatorname{Aut}_{\boldsymbol{k}} K$

Proof. Notice that the field extensions $\boldsymbol{k} \subseteq K$ and $E \subseteq L$ are both Galois. Let $u_1, \dots, u_n$ be the roots of f in L. Observe that $L = E[u_1, \dots, u_n]$ and $K = \boldsymbol{k}[u_1, \dots, u_n]$. We demonstrate that $\phi : \operatorname{Aut}_E L \to \operatorname{Aut}_{\boldsymbol{k}} K$, given by $\phi(\sigma) = \sigma|_K$, is an injective group homomorphism. It is clear that $\sigma|_K : K \to L$ is a homomorphism of fields over $\boldsymbol{k}$. On the other hand, σ permutes the roots of f; hence, $\sigma(K) \subseteq K$. The homomorphism $\sigma|_K$ is injective because all homomorphisms of fields are injective and $\dim_{\boldsymbol{k}} K = \dim_{\boldsymbol{k}} \sigma(K)$. Thus, $\sigma|_K$ is an element of $\operatorname{Aut}_{\boldsymbol{k}} K$. The map ϕ is injective because each element σ of $\operatorname{Aut}_E L$ is completely determined by the action of σ on $\{u_1, \dots, u_n\}$ since $L = E[u_1, \dots, u_n]$. □

Remark. In the set up of Lemma 5.75, it often will happen that $\operatorname{Aut}_E L$ is isomorphic to a proper subgroup of $\operatorname{Aut}_{\boldsymbol{k}} K$. Indeed, if some u_i is in E, but not K, then $\dim_E L < \dim_{\boldsymbol{k}} K$ (hence, $|\operatorname{Aut}_E L| < |\operatorname{Aut}_{\boldsymbol{k}} K|$.) In the most extreme case, if $E = K$, then L would also be K and $\operatorname{Aut}_E L = \langle \mathrm{id} \rangle$, even though $\operatorname{Aut}_{\boldsymbol{k}} K$ could be arbitrarily complicated.

Lemma 5.76. *Let $\boldsymbol{k} \subseteq K$ be a Galois field extension with $\dim_{\boldsymbol{k}} K = p$ for some prime integer p. Suppose*[60] *that $\boldsymbol{k}$ contains an element ω with $\omega^p = 1$ and $\omega \neq 1$. Then $K = \boldsymbol{k}[d]$, for some $d \in K$ with $d^p \in \boldsymbol{k}$.*

Proof. It is clear that if $u \in K \setminus \boldsymbol{k}$, then $K = \boldsymbol{k}[u]$ (because $\dim_{\boldsymbol{k}} \boldsymbol{k}[u]$ divides p and is not equal to 1). Fix one such u. We look for an element $d \in K \setminus \boldsymbol{k}$ with $d^p \in \boldsymbol{k}$. The Galois group $\operatorname{Aut}_{\boldsymbol{k}} K$ is cyclic of order p; hence $\operatorname{Aut}_{\boldsymbol{k}} K = \langle \sigma \rangle$ for some automorphism σ. Let d be the following element of K:

$$d = \sum_{i=0}^{p-1} \omega^i \sigma^i(u).$$

Observe that

$$\omega\sigma(d) - d = \omega^p \sigma^p(u) - \omega^0 \sigma^0(u) = u - u = 0.$$

Thus

$$\omega\sigma(d) = d$$

and

$$\sigma(d) = \tfrac{d}{\omega}.$$

Observe that $\sigma(d) \neq d$; but $\sigma(d^p) = \Big(\sigma(d)\Big)^p = \Big(\frac{d}{\omega}\Big)^p = d^p$. Thus $\operatorname{Aut}_{\boldsymbol{k}} K$ moves d; but $\operatorname{Aut}_{\boldsymbol{k}} K$ leaves d^p fixed. The extension $\boldsymbol{k} \subseteq K$ is Galois; it follows that $d \in K \setminus \boldsymbol{k}$ and $d^p \in \boldsymbol{k}$. □

Proof of the direction ($\Leftarrow$) of Theorem 5.69.

We start with a field $\boldsymbol{k}$ of characteristic zero, a polynomial $f \in \boldsymbol{k}[x]$, and a splitting field K of f over $\boldsymbol{k}$. We assume that $\operatorname{Aut}_{\boldsymbol{k}} K$ is a solvable group. We prove that f is solvable by radical.

We first adjoin all $|\operatorname{Aut}_{\boldsymbol{k}} K|$ roots of 1 to $\boldsymbol{k}$. We are now in the situation

$$\begin{tikzcd}
 & L \\
K \arrow[ur, hook] & \\
 & E \arrow[uu, hook] \\
\boldsymbol{k} \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}$$

where $E = \boldsymbol{k}[d]$, with $d^{|\operatorname{Aut}_{\boldsymbol{k}} K|} = 1$, and L is the splitting field of f over E. We know from Lemma 5.75 that $\operatorname{Aut}_E L$ is isomorphic to a subgroup of the solvable group $\operatorname{Aut}_{\boldsymbol{k}} K$. Thus, Theorem 5.61 guarantees that $\operatorname{Aut}_E L$ is a solvable group. It suffices to exhibit a "root tower" between E and L.

The group $\operatorname{Aut}_E L$ is solvable; so there exist subgroups

$$\langle \mathrm{id}\rangle = H_s \lhd H_{s-1} \lhd \cdots \lhd H_1 \lhd H_0 = \operatorname{Aut}_E L \tag{5.76.1}$$

with $\frac{H_i}{H_{i+1}}$ Abelian for all i. We may refine any such chain of subgroups in order to insist that in the ultimate chain, $\frac{H_i}{H_{i+1}}$ is a cyclic group of prime order p_i for each i. Arrange the notation in order to assume that (5.76.1) has this property.

Consider the fields

$$L^{H_i} \subseteq L^{H_{i+1}} \subseteq L.$$

The extension $L^{H_{i+1}} \subseteq L$ is Galois with Galois group H_{i+1}; the extension

$$L^{H_i} \subseteq L$$

is Galois with Galois group H_i; and $H_{i+1} \lhd H_i$. Apply Theorem 5.20.(d) to conclude that the field extension $L^{H_i} \subseteq L^{H_{i+1}}$ is Galois. We already know that this field extension has dimension p_i for some prime integer p_i and that L^{H_i} contains p_i distinct ${p_i}^{\text{th}}$-roots of 1. Apply Lemma 5.76 to see that $L^{H_{i+1}} = L^{H_i}[d_i]$ for some d_i with $d_i^{p_i} \in L^{H_i}$. □

5.I. **Polynomials with Galois group S_n, Part 1.**

In this section we prove Theorem 5.83 and we show Example 5.82.

Theorem. 5.83. *If p is a prime integer, then there exists an irreducible polynomial $f \in \mathbb{Q}[x]$ of degree p with $\operatorname{Aut}_{\mathbb{Q}} K = S_p$, where K is the splitting field of f over $\mathbb{Q}$.*

(In section 5.L we use a different technique to prove the same result for S_n, where the integer n is not required to be prime.) Of course, if $\operatorname{Aut}_{\mathbb{Q}} K = S_n$ for $5 \le n$, and K is the splitting field of f over $\mathbb{Q}$, then f is not solvable by radical over $\mathbb{Q}$.

Example. 5.82. If K is the splitting field of the polynomial $f = x^5 - 16x - 2$ over $\mathbb{Q}$, then $\operatorname{Aut}_{\mathbb{Q}} K$ is equal to S_5.

In this section f is an irreducible polynomial in $\mathbb{Q}[x]$, $r_1, \dots, r_n$ are the roots of f in $\mathbb{C}$, and $K = \mathbb{C}[r_1, \dots, r_n]$. Recall that $\operatorname{Aut}_{\mathbb{Q}} K$ is isomorphic to a subgroup of the group of permutations of $\{r_1, \dots, r_n\}$. (If $\sigma \in \operatorname{Aut}_{\mathbb{Q}} K$, then $\sigma|_{\{r_1,\dots,r_n\}}$ is a permutation; furthermore, each automorphism in $\operatorname{Aut}_{\mathbb{Q}} K$ is completely determined by the behavior of the automorphism on $\{r_1, \dots, r_n\}$.) There are two types of calculation in the section. We identify small subsets of S_n that generate all of S_n and we do calculus on particular polynomials in $\mathbb{Q}[x]$.

First we do some fiddling with permutations.

Doodle 5.77. Let $\{i_1, \dots, i_n\} = \{1, \dots, n\}$. If G is a subgroup of S_n and

$$\sigma = (i_1, i_2) \quad \text{and} \quad \rho = (i_1, i_2, \dots, i_n)$$

are elements of G, then $G = S_n$.

Proof. First observe that $(i_j, i_{j+1}) \in G$ for all j with $1 \le j \le n-1$. Indeed,

$$\begin{aligned} \rho\sigma\rho^{-1} &= (i_2, i_3), \\ \rho(i_2, i_3)\rho^{-1} &= (i_3, i_4), \\ &\vdots \\ \rho(i_{n-2}, i_{n-1})\rho^{-1} &= (i_{n-1}, i_n), \end{aligned}$$

are all in G. Then observe that $(i_1, i_j) \in G$ for $2 \le j \le n$. This assertion also follows by induction. We start with (i_1, i_2) in G. If $3 \le j \le n$ and $(i_1, i_{j-1}) \in G$, then

$$(i_1, i_j) = (i_1, i_{j-1})(i_{j-1}, i_j)(i_1, i_{j-1}) \in G.$$

Finally, observe that every transposition (a, b) in S_n is in G. Indeed, if a, b, and i_1 are distinct elements of $\{1, \dots, n\}$, then

$$(a, b) = (i_1, a)(i_1, b)(i_1, a) \in G.$$

Every element of S_n is a product of transpositions; see section 2.F. □

Observation 5.78. *Let G be a subgroup of S_n which contains an n-cycle and a transposition.*

(a) *If n is prime then $G = S_n$.*

(b) *If G also contains an $(n-1)$ cycle, then $G = S_n$.*

Example 5.79. The hypothesis that n is prime is needed in Observation 5.78.(a). Indeed,

$$\langle (1234), (24) \rangle$$

is equal to $D_4 \subsetneq S_4$.

Roughly speaking, the reason the proof of Observation 5.78 does not apply to Example 5.79 is that the square of a 4-cycle is not a 4-cycle. On the other hand, if p is a prime integer, then every non-trivial power of a p-cycle is a p-cycle; see Remark 5.80

Remark 5.80. Let ρ be a p-cycle in S_n for some prime integer p and some integer n with $p \le n$. Fix an exponent i with $1 \le i \le p-1$. Lagrange's Theorem ensures that ρ^i has order p. If ρ^i is equal to a product of disjoint cycles, then p (which is the order of ρ^i) is the least common multiple of the cycle lengths. It follows that ρ^i is also a p-cycle.

Proof of Observation 5.78

(a) In this part of the proof we write p for the prime integer n. Let $\sigma = (i_1, i_2)$ be a transposition in G. The hypothesis also ensures that G contains a p-cycle. Each non-identity power of the p-cycle is a p-cycle (see Remark 5.80 if necessary) and the group generated by the p-cycle acts on $\{1, \dots, p\}$ in a transitive manner.[65] Thus, some power of the p-cycle carries i_1 to i_2. In particular, there is a p-cycle in G of the form $\rho = (i_1, i_2, i_3, \dots, i_p)$. Apply Doodle 5.77 to see that $G = S_p$.

[65] The group G acts transitively on the set S if for every pair of elements s_1 and s_2 of S, there is an element $g \in G$ with $g(s_1) = s_2$.

(b) The hypothesis ensures that G contains an $(n-1)$-cycle $(i_1, \dots, i_{n-1})$ and a transposition (i_a, i_b), where

$$i_a, i_b \in \{i_1, \dots, i_{n-1}, i_n\} = \{1, 2, \dots, n\}.$$

We first modify the transposition, if necessary, in order to obtain a transposition in G which moves i_n. The hypothesis guarantees that G contains an n-cycle. Thus, G acts transitively on $\{1, \dots, n\}$. If $i_n \notin \{i_a, i_b\}$, then there exists an element σ of G with $\sigma(i_a) = i_n$. Let c be the index with $\sigma(i_b) = i_c$. Notice that $1 \le c \le n-1$ and that

$$(i_n, i_c) = \sigma(i_a, i_b)\sigma^{-1} \in G.$$

Observe that

$$(i_n, i_c, i_{c+1}, \dots, i_{n-1}, i_1, i_2, \dots, i_{c-1}) = (i_n, i_c)(i_1, i_2, \dots, i_{n-1})$$

is an n-cycle in G. Apply Doodle 5.77 to the pair of elements

$$(i_n, i_c) \quad \text{and} \quad (i_n, i_c, i_{c+1}, \dots, i_{n-1}, i_1, i_2, \dots, i_{c-1})$$

in G in order to conclude that $G = S_n$. □

Theorem 5.81. *Let f be an irreducible polynomial in $\mathbb{Q}[x]$ with $\deg f = p$ for some prime integer p. If f has exactly two non-real roots, then $\operatorname{Aut}_{\mathbb{Q}} K = S_p$, where K is the splitting field of f in $\mathbb{C}$.*

Proof. The polynomial f is irreducible so p, which is equal to $\deg f$, divides[66] $\dim_{\mathbb{Q}} K = |\operatorname{Aut}_{\mathbb{Q}} K|$. Apply Cauchy's Theorem 2.77 (or the Sylow Theorems) to see that $\operatorname{Aut}_{\mathbb{Q}} K$ contains an element ρ of order p. When one views ρ as a permutation of the p roots of f in K, ρ is a p-cycle. Complex conjugation is another element of $\operatorname{Aut}_{\mathbb{Q}} K$. It is clear that complex conjugation exchanges the two non-real roots of f and leaves the real roots fixed. Thus, $\operatorname{Aut}_{\mathbb{Q}} K$ also contains a transposition. Apply Observation 5.78 to conclude that $\operatorname{Aut}_{\mathbb{Q}} K = S_p$. □

Example 5.82. If K is the splitting field of the polynomial $f = x^5 - 16x - 2$ over $\mathbb{Q}$, then $\operatorname{Aut}_{\mathbb{Q}} K$ is equal to S_5.

Proof. Apply Eisenstein's Criteria to see that f is irreducible. Apply calculus to see that f has exactly three real roots. In particular, f' has exactly two real roots; f is an increasing function on the open intervals $(-\infty, \frac{-2}{\sqrt[4]{5}})$ and $(\frac{2}{\sqrt[4]{5}}, \infty)$; and f is a decreasing function on the open interval $(\frac{-2}{\sqrt[4]{5}}, \frac{2}{\sqrt[4]{5}})$. Thus, f has at most 3 real roots. On the other hand,

$$f(-2) = -2, \quad f(-1) = 13, \quad f(0) = -2, \quad f(3) = 193;$$

hence f has at least 3 real roots by the Intermediate Value Theorem. Apply Theorem 5.81 to conclude $\operatorname{Aut}_{\mathbb{Q}} K$ is equal to S_5. □

Theorem 5.83. *If p is a prime integer, then there exists an irreducible polynomial $f \in \mathbb{Q}[x]$ of degree p with $\operatorname{Aut}_{\mathbb{Q}} K = S_p$, where K is the splitting field of f over $\mathbb{Q}$.*

[66] If u is a root of f in K, then $\dim_{\mathbb{Q}} \mathbb{Q}[u] = p$. Now use Observation 5.3.

Proof. The assertion is obvious when $p = 2$. We take p to be at least 3. Let m and

$$n_1 < n_2 < \cdots < n_{p-2}$$

be even integers. The n's can be arbitrary. We will pick m later; at this point we merely insist that $2 \le m$. Let

$$g(x) = (x^2 + m)(x - n_1) \cdots (x - n_{p-2}).$$

Observe that

$$2 < |g(r)| \text{ for every odd integer } r.$$

Let $f = g - 2$. Use Eisenstein's Criteria to see f is irreducible. Observe that f has at least $p - 2$ real roots. (Indeed, g has exactly $p - 2$ real roots. We have shifted the graph a little bit. The new graph still crosses the x-axis at least $p - 2$ times. I will write an argument with more details; but the idea of shifting a graph by a little bit is the idea.) Observe that $f(n_i) = -2 < 0$ for all i and $0 < f(n_i + 1)$ for all ODD i. Apply the Intermediate Value Theorem to see that f has a root in each of the closed intervals:

$$[n_1, n_1{+}1],\ [n_1{+}1, n_2],\ [n_3, n_3{+}1],\ [n_3{+}1, n_4],\ \dots,\ [n_{p-4}, n_{p-4}{+}1],\ [n_{p-4}{+}1, n_{p-3}],\ [n_{p-2}, n_{p-2}{+}1].$$

Thus, f has at least $p - 2$ real roots. We choose m large enough to guarantee that f has at least one non-real root.[67] We do this by forcing $\sum_i r_i^2 < 0$ where $r_1, \dots, r_p$ are the roots of f in $\mathbb{C}$. On the one hand, we expand the original formulation of f to see that

$$f = x^p - \left(\sum_{i=1}^{p-2} n_i\right) x^{p-1} + \left(m + \sum_{1 \le i < j \le p-2} n_i n_j\right) x^{p-2} + \dots.$$

On the other hand,

$$f = \prod_{i=1}^{p} (x - r_i) = x^p - \left(\sum_{i=1}^{p} r_i\right) x^{p-1} + \left(\sum_{1 \le i < j \le p} r_i r_j\right) x^{p-2} + \dots.$$

Thus,

$$\sum_{i=1}^{p} r_i = \sum_{i=1}^{p-2} n_i \quad \text{and} \quad m + \sum_{1 \le i < j \le p-2} n_i n_j = \sum_{1 \le i < j \le p} r_i r_j.$$

We want to choose m large enough to guarantee that

$$\sum_{i=1}^{p} r_i^2 < 0.$$

We know that

$$\sum_{i=1}^{p} r_i^2 = \left(\sum_{i=1}^{p} r_i\right)^2 - 2 \sum_{1 \le i < j \le p} r_i r_j = \left(\sum_{i=1}^{p-2} n_i\right)^2 - 2\left(m + \sum_{1 \le i < j \le p-2} n_i n_j\right) = \sum_{i=1}^{p-2} n_i^2 - 2m.$$

If

$$\frac{1}{2} \sum_{i=1}^{p-2} n_i^2 < m,$$

[67] Of course, if z is a non-real root, then the complex conjugate $\bar{z}$ is a different non-real root.

(and $2 \le m$), then f has exactly $p-2$ real roots and Theorem 5.81 guarantees that $\operatorname{Aut}_{\mathbb{Q}} K = S_p$, where K is the splitting field of f over K. □

5.J. **Finite fields.** We prove the following Theorem.

Theorem 5.84.

(a) *If $\boldsymbol{k}$ is a finite field of characteristic p, then $|\boldsymbol{k}| = p^n$ for some n.*
(b) *For each positive integer n, and each positive prime integer p, there exists a field $\boldsymbol{k}$, with $|\boldsymbol{k}| = p^n$. (The field $\boldsymbol{k}$ is called the "Galois field of order p^n and is denoted* $\mathrm{GF}(p^n)$.)
(c) *If $\boldsymbol{k} \subseteq K$ and $\boldsymbol{k} \subseteq K'$ are finite fields with $\dim_{\boldsymbol{k}} K = \dim_{\boldsymbol{k}} K' = n$, then K and K' are isomorphic over $\boldsymbol{k}$. (In particular,* $\mathrm{GF}(p^n)$ *is unique. It is the splitting field of $x^{p^n} - x$ over $\frac{\mathbb{Z}}{(p)}$.)*
(d) *If $\boldsymbol{k} \subseteq K$ are finite fields, then $K = \boldsymbol{k}[\alpha]$ for some α and the extension is Galois with a cyclic Galois group.*
(e) *For all positive integers n and all finite fields $\boldsymbol{k}$, there is a polynomial $f \in \boldsymbol{k}[x]$ such that* $\deg f = n$ *and f is irreducible.*

Proof.
(a) The field $\boldsymbol{k}$ is a finite dimensional vector space over the field $\frac{\mathbb{Z}}{(p)}$. If $\dim_{\frac{\mathbb{Z}}{(p)}} \boldsymbol{k} = n$, then $\boldsymbol{k}$ has exactly p^n elements.

(b) Let $\widetilde{K}$ be the splitting field of $x^{p^n} - x$ over $\frac{\mathbb{Z}}{(p)}$, and let K be the set of roots of $x^{p^n} - x$ in $\widetilde{K}$. Observe that the elements of K form a field.[68] Observe further that $x^{p^n} - x$ has distinct roots in $\widetilde{K}$ because $x^{p^n} - x$ and $(x^{p^n} - x)' = -1$ have no factors in common. Thus, K is a field with exactly p^n elements.

(c) Let $\boldsymbol{k} \subseteq K$ be finite fields with $|\boldsymbol{k}| = p^m$ and $\dim_{\boldsymbol{k}} K = n$. We prove that K is the splitting field of $x^{p^{nm}} - x$ over $\boldsymbol{k}$. This assertion implies everything stated in (c) because of Theorem 5.35. In fact, we prove that every element of K satisfies $x^{p^{nm}} - x$. Surely, 0 satisfies this equation. Furthermore, $(K \setminus \{0\}, \times)$ is a multiplicative group of order $p^{nm} - 1$ (because K has p^{nm} elements); so if $\alpha \in K$ with $\alpha \neq 0$, then $\alpha^{p^{nm}-1} = 1$ and $\alpha^{p^{mn}} = \alpha$.

(d) We are given $\frac{\mathbb{Z}}{(p)} \subseteq \boldsymbol{k} \subseteq K$ with $|\boldsymbol{k}| = p^m$ and $|K| = p^{nm}$. We want to prove that $K = \boldsymbol{k}[\alpha]$ for some α and that $\boldsymbol{k} \subseteq K$ is a Galois extension with a cyclic Galois group.

Well, $K^* = (K \setminus \{0\}, \times)$ is a cyclic multiplicative group of order $p^{nm} - 1$. Let α be a generator of this group. Observe that $K = \boldsymbol{k}[\alpha]$. Let $\sigma : K \to K$ be the homomorphism[69] $\sigma(u) = u^p$ for all $u \in K$. Observe that $K^{\langle\sigma\rangle} = \frac{\mathbb{Z}}{(p)}$. (Every element of $\frac{\mathbb{Z}}{(p)}$ satisfies $x^p - x = 0$ (hence every element of $\frac{\mathbb{Z}}{(p)}$ is fixed by σ) and at most p elements of the field K can be roots of a polynomial of degree p.) Thus, $K^{\langle\sigma\rangle} = \frac{\mathbb{Z}}{(p)}$; $\frac{\mathbb{Z}}{(p)} \subseteq K$ is a Galois extension; and $\operatorname{Aut}_{\frac{\mathbb{Z}}{(p)}} K = \langle\sigma\rangle$. (See Remark 5.85, if

[68] Let $q = p^n$. Suppose that x and y are in K; so $x^q = x$ and $y^q = y$. Observe that $(xy)^q = x^q y^q = xy$ and $(x+y)^q = x^q + y^q = x + y$. The formula for addition works because the characteristic of K is equal to p.

[69] Algebraists and Number Theorists call this the Frobenius Homomorphism.

necessary.) The automorphism group $\operatorname{Aut}_{\frac{\mathbb{Z}}{(p)}} K$ is cyclic. Apply Theorem 5.20 to the fields

$$\frac{\mathbb{Z}}{(p)} \subseteq \boldsymbol{k} \subseteq K$$

in order to conclude that $\boldsymbol{k} \subseteq K$ is a Galois extension with automorphism group a subgroup of the cyclic group $\operatorname{Aut}_{\frac{\mathbb{Z}}{(p)}} K$. Every subgroup of a cyclic group is cyclic.

(e) Let $\boldsymbol{k} = \mathrm{GF}(p^m)$ and let K be the splitting field of $x^{p^{nm}} - x$ over k. We know from (d) that $K = \boldsymbol{k}[\alpha]$ for some α and that $\dim_{\boldsymbol{k}} K = \log_{|\boldsymbol{k}|} |K| = n$. The minimal polynomial of α over $\boldsymbol{k}$ is irreducible in $\boldsymbol{k}[x]$ and has degree n.

□

The next Fact should appear much earlier. I was not able to find it, if we wrote it down. I don't think it is fair to hide it somewhere you have already studied; so I will just put it here.

Fact 5.85. *Let K be a field and G be a finite group of automorphisms of K. Then the field extension $K^G \subseteq K$ is Galois and* $\operatorname{Aut}_{K^G} K = G$.

Proof. We proved in Remark 5.15 that $K^G \subseteq K$ is Galois. We still must nail down the assertion about $\operatorname{Aut}_{K^G} K$. We proved in Lemma 5.26 that $\dim_{K^G} K \le |G|$. It follows that $K^G \subseteq K$ is a finite dimensional Galois extension; consequently, Theorem 5.20 may be applied to this field extension. The proof of Theorem 5.20 shows that if $\boldsymbol{k} \subseteq K$ is a finite dimensional Galois extension and $\boldsymbol{H}$ is a subgroup of $\operatorname{Aut}_{\boldsymbol{k}} K$, then $\operatorname{Aut}_{K^H} K = \boldsymbol{H}$. In the present situation, $K^G \subseteq K$ is a finite dimensional Galois extension, and $\boldsymbol{G}$ is a subgroup of $\operatorname{Aut}_{K^G} K$; and therefore, $\operatorname{Aut}_{K^G} K = \boldsymbol{G}$. □

5.K. **Cyclotomic extensions (Regular *n*-gons).** We already know the following three results.

Lemma. 5.10. *If $u \in \mathbb{C}$, then u is constructible by ruler and compass if and only if there exist $u_1, \dots, u_n$ in $\mathbb{C}$ with $u \in \mathbb{Q}[u_1, \dots u_n]$ and $u_i^2 \in \mathbb{Q}[u_1, \dots u_{i-1}]$.*

Theorem. 5.6. *If u is a complex number which is constructible by ruler and compass, then* $\dim_{\mathbb{Q}} \mathbb{Q}[u] = 2^n$ *for some n.*

Corollary. 5.13. *If p is a prime integer and a regular p-gon is constructible using ruler and compass, then p is a Fermat prime*; *that is, $p = 2^{2^t} + 1$ for some t.*

In this section we learn the following four results.

Theorem 5.86. *If $u \in \mathbb{C}$, then u is constructible by ruler and compass if and only if* $\dim_{\mathbb{Q}} K = 2^t$, *for some t, where K is the Galois closure of* $\mathbb{Q}(u)$. (*In other words, K is the splitting field of the minimal polynomial of u over* $\mathbb{Q}$.)

Remark 5.87. Theorem 5.86 shows that the necessary condition of Theorem 5.6 is not sufficient. Think of any irreducible polynomial f of degree 4 with $\operatorname{Aut}_{\mathbb{Q}} K = S_4$, where K is the splitting field of f. If u is a root of f in K, then $\dim_{\mathbb{Q}} \mathbb{Q}(u) = 4$ but u is not constructible.

Theorem 5.88. *The minimal polynomial of $e^{(2\pi \mathbb{i})/n}$ has degree $\phi(n)$, where $\phi(n)$ is the Euler ϕ-function.*[70] *Furthermore, the minimal polynomial*[71] *of $e^{(2\pi \mathbb{i})/n}$ over $\mathbb{Q}$ is*

$$\Phi_n(x) = \prod_r (x - e^{(2\pi \mathbb{i} r)/n}),$$

where the product is taken over

$$\{r \in \{1, \dots, n\} \mid r \text{ and } n \text{ are relatively prime}\}.$$

Corollary 5.89. *A regular n-gon is constructible by ruler and compass if and only if $n = 2^e p_2 \cdots p_s$, where $p_1, \dots, p_s$ are distinct Fermat primes.*[72]

Corollary 5.89 is a quick consequence of Theorems 5.86 and 5.88, together with two elementary properties of the Euler ϕ-function.

Observation 5.90. *Let ϕ represent the Euler ϕ-function.*

(a) *If a and b are relatively prime integers, then $\phi(ab) = \phi(a)\phi(b)$.*
(b) *If p is a prime integer and e is a positive integer, then $\phi(p^e) = p^{e-1}(p-1)$.*

Proof. (a) The Chinese Remainder Theorem gives an isomorphism of rings

$$\frac{\mathbb{Z}}{(ab)} \longrightarrow \frac{\mathbb{Z}}{(a)} \oplus \frac{\mathbb{Z}}{(b)}.$$

Thus, there is a one-to-one correspondence between the set of multiplicative units of the ring $\frac{\mathbb{Z}}{(ab)}$

$$= \{\bar{n} \mid n \text{ and } ab \text{ are relatively prime}\}$$

and the set of units of the ring $\frac{\mathbb{Z}}{(a)} \oplus \frac{\mathbb{Z}}{(b)}$

$$= \{(\bar{n}, \bar{m}) \mid n \text{ and } a \text{ are relatively prime, and } m \text{ and } b \text{ are relatively prime}\}.$$

(b) Observe that all of the integers between 1 and p^e are relatively prime to p^e except $p, \dots, p^e$. It follows that

$$\phi(p^e) = p^e - p = p^{e-1}(p-1). \qquad \square$$

Assume Theorems 5.86 and 5.88 for the time being and prove Corollary 5.89. We know from Theorem 5.88 that $\mathbb{Q}(e^{(2\pi \mathbb{i})/n})$ is the splitting field of the minimal polynomial of $e^{(2\pi \mathbb{i})/n}$ over Q and that $\dim_{\mathbb{Q}} \mathbb{Q}(e^{(2\pi \mathbb{i})/n})$ is equal to $\phi(n)$, where ϕ is the Euler ϕ-function. Thus,

$$\dim_{\mathbb{Q}}(\text{the splitting field of the minimal polynomial of } e^{(2\pi \mathbb{i})/n} \text{ over } \mathbb{Q}) = \phi(n) \tag{5.90.1}$$

A regular n-gon is constructible by ruler and compass if and only if $e^{(2\pi \mathbb{i})/n}$ is constructible by ruler and compass. Apply Theorem 5.86 and (5.90.1) to see that $e^{(2\pi \mathbb{i})/n}$ is constructible by ruler and compass if and only if $\phi(n) = 2^t$ for some t.

Let n be an arbitrary integer with $3 \le n$. Write n in the form $n = 2^e p_2^{e_2} \cdots p_s^{e_s}$, where e is a non-negative integer, the p_i are distinct odd prime integers, and e_i is a positive integer for $2 \le i \le s$. We

[70] The Euler ϕ-function of n is the number of integers in the set $\{1, 2, \dots, n\}$ which are relatively prime to n.

[71] The polynomial $\Phi_n(x)$ is called the n^{th}-cyclotomic polynomial.

[72] This result was promised in Remark (b) after Corollary 5.13.

have seen that a regular n-gon is constructible by ruler and compass if and only if $\phi(2^e p_2^{e_2} \cdots p_s^{e_s}) = 2^t$ for some t. Thus, a regular n-gon is constructible by ruler and compass if and only if $n = 2^e p_2^{e_2} \cdots p_s^{e_s}$ and

$$2^{e-1}(2-1)p_2^{e_2-1}(p_2-1)p_3^{e_3-1}(p_3-1)\cdots p_s^{e_s-1}(e_s-1) = 2^t.$$

A regular n-gon is constructible by ruler and compass if and only if $n = 2^e p_2^{e_2} \cdots p_s^{e_s}$ and $0 \le e$ is arbitrary, $e_2 = \cdots = e_2 = 1$, and $p_i - 1$ is a power of 2, for $2 \le i \le s$. We saw in the proof of Corollary 5.13 that if a prime integer is equal to $2^a + 1$ for some positive integer a, then a must be a power of 2 and therefore the prime integer $2^a + 1$ must be a Fermat prime. We conclude that a regular n-gon is constructible by ruler and compass if and only if $n = 2^e p_2 \cdots p_s$ for distinct Fermat prime integers p_i and some non-negative power e. □

We now prove

Theorem. 5.86. *If $u \in \mathbb{C}$, then u is constructible by ruler and compass if and only if* $\dim_{\mathbb{Q}} K = 2^t$, *for some t, where K is the splitting field of the minimal polynomial of u over $\mathbb{Q}$.*

($\Leftarrow$) The extension $\mathbb{Q} \subseteq K$ is Galois with a Galois group of order 2^t. Use the Sylow theorems to get a chain of subgroups of $\operatorname{Aut}_{\mathbb{Q}} K$ with the property that each subgroup has index two in the next subgroup in the chain. Use Galois theory to produce a chain of fields from

$$\mathbb{Q} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_t = K,$$

with $\dim_{K_{i-1}} K_i = 2$. If u_i is any element in $K_i \setminus K_{i-1}$ and

$$f_i = x^2 + b_i x + c_i$$

is the minimal polynomial of u_i over K_{i-1}, then $K_i = K_{i-1}\left[\sqrt{b_i^2 - 4c_i}\right]$. Notice that if u_i' is the other root of $f_i \in \mathbb{C}$, then

$$b_i = -(u_i + u_i'), \quad c_i = u_i u_i', \quad \text{and} \quad (u_i - u_i')^2 = b_i^2 - 4c.$$

It is easy to see that $K_{i-1}[u_i - u_i'] = K_{i-1}[u_i]$. Indeed

$$u_i = \frac{1}{2}\Big((u_i - u_i') + \underbrace{(u_i + u_i')}_{\in K_{i-1}}\Big).$$

Apply Lemma 5.10 to see that every element of K is constructible by ruler and compass. In particular, u is constructible by ruler and compass.

($\Rightarrow$) We are given $u_1, \ldots, u_n \in \mathbb{C}$ with $u_i^2 \in \mathbb{Q}[u_1, \ldots, u_{i-1}]$. We hope to show that there exists a Galois extension $\mathbb{Q} \subseteq L$ such that $\dim_{\mathbb{Q}} L$ is equal to 2 to a power and $\mathbb{Q}[u_1, \ldots, u_n] \subseteq L$. Of course, if we have a particular u in our hands with $u \in \mathbb{Q}[u_1, \ldots, u_n]$, then the splitting field of the minimal polynomial of u over $\mathbb{Q}$ is contained in L and therefore the dimension of this splitting field as a vector space over $\mathbb{Q}$ is also 2 to a power.

We do the usual thing. Let f_i be the minimal polynomial of u_i over $\mathbb{Q}$ and let L the the splitting field of $\prod_{i=1}^n f_i$ over $\mathbb{Q}$. It is clear that $\mathbb{Q} \subseteq L$ is a Galois extension with $\mathbb{Q}[u_1, \ldots, u_n] \subseteq L$. We prove that there is a chain of fields

$$\mathbb{Q} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_s = L$$

so that

$$\dim_{L_{i-1}} L_i \in \{1, 2\}$$

for all i. Let $\operatorname{Aut}_{\mathbb{Q}} L = \{\sigma_1 = \mathrm{id}, \sigma_2, \dots, \sigma_N\}$. Observe that

$$L = \mathbb{Q}[u_1, \dots u_n, \sigma_2(u_1), \dots \sigma_2(u_n), \dots, \sigma_N(u_1), \dots \sigma_N(u_n)]$$

and

- $u_i^2 \in \mathbb{Q}[u_1, \dots, u_{i-1}]$, for $1 \le i \le n$,
- $(\sigma_2 u_1)^2 = \sigma_2(u_1^2) \in \mathbb{Q}[u_1, \dots, u_n]$, because $u_1^2 \in \mathbb{Q}$ and σ_2 leaves $\mathbb{Q}$ fixed,
- $(\sigma_2 u_2)^2 = \sigma_2(u_2^2) \in \mathbb{Q}[u_1, \dots, u_n, \sigma_2(u_1)]$, because $u_2^2 \in \mathbb{Q}[u_1]$ and σ_2 leaves $\mathbb{Q}$ fixed,
- etc.

We conclude that $\dim_Q L$ is equal to 2 to a power and the proof is complete. □

We conclude this section by proving

Theorem. 5.88. *The cyclotomic polynomial*

$$\Phi_n(x) = \prod_r (x - e^{(2\pi i r)/n}),$$

where the product is taken over

$$\{r \in \{1, \dots, n\} \mid r \textit{ and } n \textit{ are relatively prime}\},$$

is in $\mathbb{Z}[x]$ *and is irreducible in* $\mathbb{Z}[x]$.

Examples 5.91.

$$\begin{aligned}
\Phi_1(x) &= x - 1 \\
\Phi_2(x) &= \frac{x^2 - 1}{\Phi_1(x)} = x + 1 \\
\Phi_3(x) &= \frac{x^3 - 1}{\Phi_1(x)} = x^2 + x + 1 \\
\Phi_4(x) &= \frac{x^4 - 1}{\Phi_1(x)\Phi_2(x)} = \frac{x^4 - 1}{x^2 - 1} = x^2 + 1 \\
\Phi_5(x) &= \frac{x^5 - 1}{\Phi_1(x)} = x^4 + x^3 + x^2 + x + 1 \\
\Phi_6(x) &= \frac{x^6 - 1}{\Phi_1(x)\Phi_2(x)\Phi_3(x)} = \frac{x^6 - 1}{(x + 1)(x^3 - 1)} = \frac{x^3 + 1}{x + 1} = x^2 - x + 1.
\end{aligned}$$

The proof of Theorem 5.88 involves a wonderful application of the Frobenius homomorphism. We prove a characteristic zero result by passing to characteristic p and applying the Frobenius homomorphism $a \mapsto a^p$. This homomorphism is magical because it makes raising to a power become a linear map – just the way our calculus students would like it to be: $(a+b)^p = a^p + b^p$. (The coefficient of $a^i b^{p-i}$ is $\binom{p}{i}$ which is divisible by p for $1 \le i \le p-1$.)

Number Theorists have always been aware of the power of the Frobenius. In the last fifty years or so, Algebraic Geometers have proved results about integration on complex manifolds by reducing the questions to rings that are finitely generated over the integers and then reducing mod p and applying the Frobenius homomorphism.

Proof. We first show that $\Phi_n(x)$ is in $\mathbb{Q}[x]$. Let g represent the polynomial

$$g_n = \prod_{\substack{d|n \\ d \neq n}} \Phi_d(x).$$

We know by induction that $g_n \in \mathbb{Z}[x] \subseteq \mathbb{Q}[x]$. Observe that

$$x^n - 1 = g_n(x)\Phi_n(x) \in \mathbb{C}[x].$$

Apply Doodle 5.92 to learn that $\Phi_n(x) \in \mathbb{Q}[x]$.

Now we show that $\Phi_n(x)$ is in $\mathbb{Z}[x]$. Write $\Phi_n(x) = \frac{a}{b} f(x)$ where a and b are relatively prime integers and $f(x)$ is a primitive[73] polynomial in $\mathbb{Z}[x]$. The polynomials $x^n - 1$ and g_n are monic polynomials in $\mathbb{Z}[x]$ hence they are also primitive. Gauss' Lemma 3.43 guarantees that the product $g_n(x)f(x)$ is a primitive polynomial in $\mathbb{Z}[x]$; so the equation

$$b(x^n - 1) = a g_n(x) f(x)$$

yields that $\frac{a}{b}$ is equal to a unit in $\mathbb{Z}$; hence $\Phi_n(x) \in \mathbb{Z}[x]$ as claimed.

Now we show that $\Phi_n(x)$ is an irreducible polynomial in $\mathbb{Z}[x]$. Suppose

$$\Phi_n(x) = h(x)k(x) \in \mathbb{Z}[x]$$

where h and k both are monic polynomials of positive degree. We look for a contradiction. Recall that if $z \in \mathbb{C}$ is a root of $\Phi_n(x)$, then z^p is also a root of $\Phi_n(x)$ for all prime integers p relatively prime to n. Identify a complex number z and a prime integer p (with p relatively prime to n) so that z is a root of $h(x)$ but z^p is not a root of $h(x)$. It follows that z^p is a root of $k(x)$.

At this point, $h(x)$ is the minimal polynomial of z in $\mathbb{Q}[x]$, but z is also a root of $k(x^p)$. It follows that $h(x)$ divides $k(x^p)$ in $\mathbb{Q}[x]$. The polynomials $h(x)$ and $k(x^p)$ both are monic polynomials in $\mathbb{Z}[x]$. Use Gauss' Lemma (3.43) again to see that $h(x)$ divides $k(x^p)$ in $\mathbb{Z}[x]$. Thus, there exists $\ell(x) \in \mathbb{Z}[x]$ with

$$k(x^p) = h(x)\ell(x) \in \mathbb{Z}[x].$$

Observe that

$$x^n - 1 = g_n(x)\Phi_n(x) = g_n(x)h(x)k(x) \in \mathbb{Z}[x].$$

Let $\bar{\ }$ represent image in $\frac{\mathbb{Z}}{(p)}$. It follows that

$$x^n - \bar{1} = \overline{g_n}(x)\bar{h}(x)\bar{k}(x) \in \frac{\mathbb{Z}}{(p)}[x].$$

[73] See Definition 3.42.

However

$$\boxed{\boxed{(\bar{k}(x))^p = \bar{k}(x^p) = \bar{h}\bar{\ell}}}$$

Here is a careful explanation of the left equality in the boxed expression. If $\alpha_i \in \frac{\mathbb{Z}}{(p)}$, then

$$\left(\sum_i \alpha_i x^i\right)^p = \sum_i \alpha_i^p x^{pi} = \sum_i \alpha_i (x^p)^i \tag{5.91.1}$$

The first equality in (5.91.1) is the fact that the Frobenius homomorphism is a homomorphism; the second equality is the fact that the Frobenius homomorphism fixes $\frac{\mathbb{Z}}{(p)}$ (that is, every element of $\frac{\mathbb{Z}}{(p)}$ satisfies $x^p - x = 0$.) At any rate, $\bar{k}$ and $\bar{h}$ are not relatively prime. (It might be useful to notice that h, k, and ℓ all are monic polynomials. Their degrees remains unchanged as one passes from $\mathbb{Z}[x]$ to $\frac{\mathbb{Z}}{(p)}[x]$.) Thus, $x^n - \bar{1}$ has repeated roots in some extension field of $\frac{\mathbb{Z}}{(p)}$. But this is a contradiction because $(x^n - \bar{1})' = \bar{n}x^{n-1}$ and $x^n - \bar{1}$ are relatively prime in $\frac{\mathbb{Z}}{(p)}[x]$ because n and p are relatively prime in $\mathbb{Z}$. □

Doodle 5.92. If $\boldsymbol{k} \subseteq K$ are fields, f and g are polynomials in $\boldsymbol{k}[x]$ with g not the zero polynomial, and h is a polynomial in $K[x]$ with $f = gh$ in $K[x]$, then $h \in \boldsymbol{k}[x]$.

Proof. The division algorithm in $\boldsymbol{k}[x]$ gives $f = gq + r$ for some polynomials q, r in $\boldsymbol{k}[x]$ with $\deg r < \deg g$. The equation $f = gq + r$ continues to hold in $K[x]$. Thus, g divides r in $K[x]$ with $\deg r < \deg g$. We conclude that r is the zero polynomial. The equation

$$gh = f = gq$$

holds in the domain $K[x]$ with g not the zero polynomial. Thus $h = q$. But $q \in \boldsymbol{k}[x]$. We conclude that $h \in \boldsymbol{k}[x]$. □

5.L. **Polynomials with Galois group S_n, Part 2: "mod p reduction".**

Theorem 5.94 was promised when we started Section 5.I. The basic idea is that we study the Galois group of a monic polynomial with integer coefficients by reducing mod p and studying the Galois group of finite field extensions.

Theorem 5.93. *Let $f \in \mathbb{Z}[x]$ be a monic polynomial, K be the splitting field of f over $\mathbb{Q}$, and let p be a prime integer. Suppose that the image $\bar{f}$ of f in $\frac{\mathbb{Z}}{(p)}[x]$ factors into distinct irreducible factors of degree*[74] *$n_1, n_2, \dots, n_r$. Then*[75] *after renumbering the roots of f, $\operatorname{Aut}_{\mathbb{Q}} K$ contains the element*

$$(1,\dots,n_1)(n_1+1,\dots,n_1+n_2)(n_1+n_2+1,\dots,n_1+n_2+n_3)\cdots\left(\left(\sum_{i=i}^{r-1} n_i\right)+1,\dots,\left(\sum_{i=i}^{r} n_i\right)\right).$$

The proof of Theorem 5.93 is given later in the section. Assume Theorem 5.93 for the time being and prove Theorem 5.94.

[74] Notice that $\sum_{i=1}^{r} n_i = \deg f$.

[75] Here is a more conceptual version of the conclusion. Think of $\operatorname{Aut}_{\mathbb{Q}} K$ is a subgroup of the group of permutations of the roots of f in K. The Theorem asserts that there is an element in $\operatorname{Aut}_{\mathbb{Q}} K$ so that when this element is written as a product of disjoint cycles it is equal to $\prod_{i=1}^{r} c_i$, where c_i is an n_i-cycle.

Theorem 5.94. *For each integer n, there exists a polynomial $f \in \mathbb{Z}[x]$ such that* $\deg f = n$ *and* $\operatorname{Aut}_{\mathbb{Q}} K = S_n$*, where K is the splitting field of f over* $\mathbb{Q}$*.*

Proof.

- Let $f_2 \in \mathbb{Z}[x]$ be a monic polynomial of degree n such that the image $\bar{f}_2$ of f_2 in $\frac{\mathbb{Z}}{(2)}[x]$ is irreducible.
- Let $f_3 \in \mathbb{Z}[x]$ be a monic polynomial of degree n such that the image $\bar{f}_3$ of f_3 in $\frac{\mathbb{Z}}{(3)}[x]$ is a linear polynomial times an irreducible polynomial of degree $n-1$.
- Let $f_5 \in \mathbb{Z}[x]$ be a monic polynomial of degree n such that the image $\bar{f}_5$ of f_5 in $\frac{\mathbb{Z}}{(5)}[x]$ is an irreducible quadratic polynomials times a product of distinct irreducible polynomials of odd degree.

Select integers $a, b, c \in \mathbb{Z}$ such that $a + b + c = 1$, $15|a$, $10|b$, and $6|c$. In particular, one can take $(a, b, c) = (-15, 10, 6)$. Observe that $\bar{g} = \bar{f}_p$ in $\frac{\mathbb{Z}}{(p)}$ for $p \in \{2, 3, 5\}$. Apply Theorem 5.93 to see that

- $\operatorname{Aut}_{\mathbb{Q}} K$ contains an n-cycle (from $p = 2$),
- $\operatorname{Aut}_{\mathbb{Q}} K$ contains an $(n-1)$-cycle (from $p = 3$),
- $\operatorname{Aut}_{\mathbb{Q}} K$ contains an element of the form $\sigma\tau$ where the two permutations are disjoint, σ is a 2-cycle, and τ has odd order (from $p = 5$).

Of course, σ and τ commute and $(\sigma\tau)^{\text{the order of } \tau} = \sigma$. Apply Observation 5.78 to complete the proof. □

The interesting mathematics that establishes Theorem 5.93 is stated as Theorem 5.95.

Theorem 5.95. *Fix a prime integer p. Let $f \in \mathbb{Z}[x]$ be a monic polynomial. Assume that the irreducible factors of the image $\bar{f}$ of f in $\frac{\mathbb{Z}}{(p)}$ are distinct.*[76] [77] *Let K be the splitting field of f over* $\mathbb{Q}$*, let $f = \prod_{i=1}^{n}(x - v_i)$ in $K[x]$, let R be the ring $R = \mathbb{Z}[v_1, \dots, v_n]$, and let L be the splitting field of $\bar{f}$ over $\frac{Z}{(p)}$. Then the following statements hold.*

(a) *There exists a ring homomorphism $\psi : R \to L$.*

(b) *If $\psi : R \to L$ is any ring homomorphism, then the restriction of ψ to the roots of f gives a bijection between the roots of f and the roots of $\bar{f}$.*

(c) *If ψ and ψ' are both ring homomorphisms from R to L, then there exists an element σ in* $\operatorname{Aut}_{\mathbb{Q}} K$ *such that $\psi' = \psi \circ \sigma$.*

Assume Theorem 5.95. Prove Theorem 5.93. Assertion (a) of Theorem 5.95 guarantees the existence of a ring homomorphism $\psi : R \to L$. Fix one such ψ for the rest of the argument. Let

[76] Notice that this hypothesis guarantees that f is a product of distinct irreducible factors in $\mathbb{Z}[x]$.

[77] A better way to say this hypothesis would be to say "Assume that the discriminant of f is not divisible by p." But if I said that, then I would have to tell you what the discriminant of a polynomial is, that the discriminant of f is zero if and only if f has repeated roots in some extension field, and that the homomorphism $\mathbb{Z} \to \mathbb{Z}/(p)$ carries the discriminant of f to the discriminant of $\bar{f}$. I could tell you all of those things, but I would rather just end the course. The words that I have written work fine and when we apply Theorem 5.95 to prove Theorem 5.93 we want to know how $\bar{f}$ factors anyhow.

$\pi : L \to L$ be the Frobenius homomorphism (that is, $\pi(\alpha) = \alpha^p$ for all $\alpha \in L$). Recall from[78] Theorem 5.84.(d) that $\operatorname{Aut}_{\mathbb{Z}/(p)} L = \langle \pi \rangle$. Observe that $\pi \circ \psi : R \to L$ is also a ring homomorphism. Apply Theorem 5.95.(c) to see that there exists an element $\sigma \in \operatorname{Aut}_{\mathbb{Q}} K$ so that $\pi \circ \psi = \psi \circ \sigma$; in other words,

$$\psi^{-1} \circ \pi \circ \psi = \sigma \tag{5.95.1}$$

is a legitimate equation on the roots of f in K. We know from the Homework problem

Homework Problem Let $\boldsymbol{k} \subseteq K$ be an extension of finite fields. Suppose that K is the splitting field of $f(x) \in \boldsymbol{k}[x]$ over $\boldsymbol{k}$. Suppose that $f = \prod_{i=1}^r f_i$ is a factorization of f into distinct irreducible factors in $\boldsymbol{k}[x]$. Prove that it is possible to number the roots of f in K so that

$$\operatorname{Aut}_{\boldsymbol{k}} K = \left\langle (1, \dots, n_1)(n_1 + 1, \dots, n_1 + n_2)(n_1 + n_2 + 1, \dots, n_1 + n_2 + n_3) \cdots \left(\left(\sum_{i=i}^{r-1} n_i \right) + 1, \dots, \left(\sum_{i=i}^{r} n_i \right) \right) \right\rangle,$$

where $n_i = \deg f_i$.

that

$$“\pi = \Big(1, \dots, n_1\Big)\Big(n_1{+}1, \dots, n_1{+}n_2\Big)\Big(n_1{+}n_2{+}1, \dots, n_1{+}n_2{+}n_3\Big) \cdots \left(\left(\sum_{i=i}^{r-1} n_i \right) + 1, \dots, \left(\sum_{i=i}^{r} n_i \right) \right).”$$

In other words, $\operatorname{Aut}_{\mathbb{Z}/(p)} L$ is cyclic and the cycle structure of the generator of $\operatorname{Aut}_{\mathbb{Z}/(p)} L$ is given by the degrees of the irreducible factors of $\bar{f}$ in $\frac{\mathbb{Z}}{(p)}[x]$. In the statement of Theorem 5.95, we labeled the roots of f in K as $v_1, \dots, v_n$. Assertion 5.95.(b) states that the roots of $\bar{f}$ in L are $\psi(v_1), \dots, \psi(v_n)$. Now we understand the roots of $\bar{f}$ in terms of the data we care about (namely the roots of f) and we can remove the “ ” from the information obtained from the HW. HW tells us that it is possible to re-number the roots $v_1, \dots, v_n$ of f so that

$$\pi = \Big(\psi(v_1), \dots, \psi(v_{n_1})\Big)\Big(\psi(v_{n_1+1}), \dots, \psi(v_{n_1+n_2})\Big), \dots$$

Apply (5.95.1) in order to learn that the element σ of $\operatorname{Aut}_{\mathbb{Q}} K$ is

$$(v_1, \dots, v_{n_1})(v_{n_1+1}, \dots, v_{n_1+n_2}) \cdots (v_{n_1+\dots+n_{r-1}+1}, \dots, v_{n_1+\dots+n_r}). \qquad \square$$

Proof of Theorem 5.95.

In Claims 5.95.2 to 5.95.4 we discover valuable information about how the ring R fits into the field K.

Claim 5.95.2. *The ring R is finitely generated as a $\mathbb{Z}$-module.*

Observe that each of the ring extensions

$$\mathbb{Z} \subseteq \mathbb{Z}[v_1] \subseteq \mathbb{Z}[v_1, v_2] \subseteq \cdots \subseteq \mathbb{Z}[v_1, \dots, v_n] = R$$

is an integral extension. Indeed, $\mathbb{Z}[v_1, \dots, v_i]$ is generated as a ring over $\mathbb{Z}[v_1, \dots, v_{i-1}]$ by v_i and v_i is integral over $\mathbb{Z}[v_1, \dots, v_{i-1}]$ because v_i satisfies the monic polynomial $f(x)$ which has coefficients in $\mathbb{Z}[v_1, \dots, v_{i-1}]$. Thus, $\mathbb{Z}[v_1, \dots, v_i]$ is generated as a $\mathbb{Z}[v_1, \dots, v_{i-1}]$-module by $1, v_i, v_i^2, \dots, v_i^{n-1}$.

[78] Maybe you have to look at the proof of the result rather than the statement.

We conclude that R is generated as an $\mathbb{Z}$-module by

$$\{v_1^{e_1}v_2^{e_2}\cdots v_n^{e_n} \mid 0 \le e_i \le n-1 \text{ for all } i\}.$$

Claim 5.95.2 has been established.

Claim 5.95.3. *There exists an integer N and elements $u_1, \dots, u_N$ in R so that $R = \mathbb{Z}u_1 \oplus \dots \mathbb{Z}u_N$.*

Indeed, every finitely generated $\mathbb{Z}$-module is the direct sum of a finite Abelian group plus a finitely generated free Abelian group.[79] Thus, as an Abelian group under addition:

$$R = G \oplus \mathbb{Z}u_1 \oplus \dots \mathbb{Z}u_N$$

for some finite subgroup G of the additive part of R and some elements $u_1, \dots, u_N$ of R. Of course, if $g \in G$, then

$$\underbrace{g + \dots + g}_{\text{the order of } G \text{ times}} = 0.$$

In other words,

$$|G| \cdot G = 0 \in R \subseteq K.$$

The field K has characteristic zero; consequently $G = 0$ and Claim 5.95.3 is established.

Claim 5.95.4. *The integer N from Claim* 5.95.3 *is equal to* $\dim_{\mathbb{Q}} K$.

It suffices to show that $K = \mathbb{Q}u_1 \oplus \dots \oplus \mathbb{Q}u_N$.

The elements $u_1, \dots u_N$ span K. Recall that $K = \mathbb{Q}[v_1, \dots, v_n]$ and $R = \mathbb{Z}[v_1, \dots, v_n]$. It follows that every element of K has the form[80]

$$\frac{\text{element of } R}{\text{non-zero element of } \mathbb{Z}} = \frac{\sum_{i=1}^N z_iu_i}{\text{non-zero element of } \mathbb{Z}},$$

for some $z_i \in \mathbb{Z}$.

The elements $u_1, \dots u_N$ are linearly independent in K. Suppose $\sum q_iu_i = 0$ with $q_i \in \mathbb{Q}$. Let b be a common denominator for $q_1, \dots, q_N$. Write q_i as $\frac{a_i}{b}$ for integers a_i.

$$0 = \sum q_iu_i = \frac{\sum a_iu_i}{b}.$$

Thus,

$$0 = \sum a_iu_i \in R = \bigoplus_{i=1}^N \mathbb{Z}u_i.$$

Conclude that each $a_i = 0$; hence each $q_i = 0$.

Claim 5.95.4 has been established.

Proof of (a). The ideal pR is not equal to the whole ring R. Indeed, as an Abelian group,

$$\frac{R}{pR} = \frac{\mathbb{Z}}{p\mathbb{Z}}u_1 \oplus \dots \oplus \frac{\mathbb{Z}}{p\mathbb{Z}}u_N.$$

[79] See Theorem 4.3 or Theorem 2.90 and near by results.

[80] Given an arbitrary element of $\mathbb{Q}[v_1, \dots, v_n]$, one gets a common denominator.

Let $\mathfrak{m}$ be a maximal ideal of R which contains pR. Notice that the kernel of the composition

$$\begin{tikzcd} \mathbb{Z} \arrow[r, hook] & R \arrow[r, two heads] & \frac{R}{\mathfrak{m}} \end{tikzcd}$$

is $p\mathbb{Z}$. Thus, $\frac{\mathbb{Z}}{p\mathbb{Z}}$ is a subfield of the field $\frac{R}{\mathfrak{m}}$. The ring R is generated over $\mathbb{Z}$ by $v_1, \dots, v_n$; hence, the field $\frac{R}{\mathfrak{m}}$ is generated over $\frac{\mathbb{Z}}{pZ}$ by the images $\bar{v}_1, \dots, \bar{v}_n$ of $v_1, \dots, v_n$ in $\frac{R}{\mathfrak{m}}$. The polynomial f has coefficients in $\mathbb{Z}$, when these coefficients are reduced mod p, we have called the result $\bar{f}$ in $\frac{\mathbb{Z}}{p\mathbb{Z}}[x]$. The elements $\bar{v}_1, \dots, \bar{v}_n$ in the field $\frac{R}{\mathfrak{m}}$ are roots of the polynomial $\bar{f}$. Thus, $\frac{R}{\mathfrak{m}}$ is a splitting field of $\bar{f}$ over $\frac{\mathbb{Z}}{pZ}$. The field L is a splitting field of $\bar{f}$ over $\frac{\mathbb{Z}}{pZ}$. According to Theorem 5.35, the fields $\frac{R}{\mathfrak{m}}$ and L are isomorphic. Fix one such isomorphism $\frac{R}{\mathfrak{m}} \xrightarrow{\cong} L$ and define $\psi : R \to L$ to be the ring homomorphism:

$$R \xrightarrow{\text{natural quotient map}} \frac{R}{\mathfrak{m}} \xrightarrow{\cong} L.$$

This completes the proof of (a).

Proof of (b). We are given a ring homomorphism $\psi : R \to L$. The field L is a field extension of $\frac{\mathbb{Z}}{(p)}$ and therefore $\psi(n) = \bar{n} \in \frac{\mathbb{Z}}{(p)}$ for all $n \in \mathbb{Z}$. In particular, the monic polynomial f in $\mathbb{Z}[x]$ is sent[81] by ψ^{ext} to the usual $\bar{f}$ in $\frac{\mathbb{Z}}{p\mathbb{Z}}[x]$. At any rate,

$$\prod(x - \bar{v}_i) = \bar{f} = \psi^{\text{ext}}(f) = \psi^{\text{ext}}\left(\prod(x - v_i)\right) = \prod(x - \psi(v_i)) \in L[x].$$

The polynomial ring $L[x]$ is a UFD. It follows that the homomorphism ψ restrictions to give a bijective map from the set $\{v_1, \dots, v_n\}$ to the set $\{\bar{v}_1, \dots, \bar{v}_n\}$. This completes the proof of (b).

Proof of (c). Here is the plan for the proof of (c). This proof is pretty clever and is due to John Tate (who passed away in October, 2019). An expanded reference maybe found at [5].

We exhibit N distinct ring homomorphisms from R to L. We consider the L-vector space $\mathscr{F}$ of functions from R to L. (This is a huge vector space because we did not put any restriction on what kinds of functions we are considering. But it truly is a vector space if θ_1 and θ_2 from R to L are functions, then $\theta_1 + \theta_2 : R \to L$ is the function which sends $r \in R$ to $\theta_1(r) + \theta_2(r) \in L$; similarly, if $\lambda \in L$, then $\lambda\theta_1$ is the function which sends $r \in R$ to $\lambda\theta_1(r) \in L$.)

Anyhow, the Dedekind Independence Theorem guarantees that any collection of **distinct** ring homomorphisms R to L is linearly independent in $\mathscr{F}$. This is an easy theorem and it is fun to prove. Of course, this Theorem tells us that the N distinct ring homomorphisms that we constructed in the first step are linearly independent.

We will show by hand that every collection of $N + 1$ ring homomorphisms $R \to L$ is a linearly dependent subset of $\mathscr{F}$. This forces every ring homomorphism $R \to L$ to be one of the N distinct ring homomorphisms we constructed in the first step.

[81] Extend the ring homomorphism $\psi : R \to L$ to the ring homomorphism $\psi^{\text{ext}} : R[x] \to L[x]$, where the restriction $\psi^{\text{ext}}|_R$ of ψ^{ext} to R is equal to ψ and $\psi^{\text{ext}}(x) = x$.

Step 1 of part (c). Fix a ring homomorphism $\psi : R \to L$. (Part (a) guarantees that such a ψ exists.) Observe that for each $\sigma \in \operatorname{Aut}_{\mathbb{Q}} K$, $\psi\circ\sigma : R \to L$ is the composition of two ring homomorphisms and therefore is ring homomorphism.[82] According to (b), the map ψ restricts to give a bijection

$$\{v_1, \dots, v_n\} \leftrightarrow \{\bar{v}_1, \dots, \bar{v}_n\};$$

therefore, the ring homomorphisms $\psi\circ\sigma : R \to L$ are distinct as σ roams over $\operatorname{Aut}_{\mathbb{Q}} K$. We have produced $N = \dim_{\mathbb{Q}} K = |\operatorname{Aut}_{\mathbb{Q}} K|$ distinct ring homomorphisms $R \to L$.

Step 2 of part (c).

Lemma. [Dedekind Independence Theorem] *Let R be a ring*[83]*, L be a field, and $\mathscr{F}$ be the L-vector space of all functions from R to L. If S is a set of distinct ring homomorphisms from R to L, then S is linearly independent in $\mathscr{F}$.*

Proof. Suppose S is linearly dependent. Let

$$\sum_{i=1}^{s} \alpha_i\chi_i = 0, \text{ with } \chi_i \in S, \alpha_i \in L \setminus \{0\}$$

be the shortest non-trivial relation on the elements of S. Of course, $s \neq 1$ because $\chi_1(1) = 1$. We will produce a shorter non-trivial relation on S. This will be a contradiction.

The ring homomorphisms χ_1 and χ_2 are different. Fix $r_0 \in R$ with $\chi_1(r_0) \neq \chi_2(r_0)$. Let r be an arbitrary element of R. Consider

$$0 = \sum_{i=1}^{s} \alpha_i\chi_i(r_0r) - \chi_1(r_0)\sum_{i=1}^{s} \alpha_i\chi_i(r) = \sum_{i=1}^{s} \alpha_i[\chi_i(r_0) - \chi_1(r_0)]\chi_i(r).$$

(We used the fact that $\chi_i(r_0r) = \chi_i(r_0)\cdot\chi_i(r)$.) It follows that

$$0 = \sum_{i=2}^{s} \alpha_i[\chi_i(r_0) - \chi_1(r_0)]\chi_i$$

is a shorter non-trivial[84] is relation on S. We have obtained a contradiction. The proof is complete. □

Step 3 of part (c). Return to the notation of Theorem 5.95. We prove that every set of $N + 1$ ring homomorphisms from R to L is a linearly dependent subset of the L-vector space $\mathscr{F}$ of all functions from R to L. We use the fact that $R = \mathbb{Z}u_1 \oplus \dots \oplus \mathbb{Z}u_N$.

[82] Please notice also that $\sigma(R) \subseteq R$, because σ permutes $\{v_1, \dots, v_n\}$ and σ acts like the identity map on $\mathbb{Z}$.

[83] Usually this result is stated when R is replaced by a group and the elements of S are group homomorphisms to the multiplicative group of L. In this case the elements of S are called group characters and are usually denoted by "χ". I notice that Jacobson states the result with R replaced by a monoid. I feel no strong inclination to look up what a monoid is. (The proof I gave used $\chi(1) = 1$ and $\chi(xy) = \chi(x)\chi(y)$. The proof I gave will work on any algebraic structure that has an identity element and a multiplication provided the maps from the algebraic structure to the field satisfy $\chi(1) = 1$ and $\chi(xy) = \chi(x)\chi(y)$.) The proof is easy and fairly versatile. (It is essentially the same argument one uses to show that non-zero eigenvectors which belong to different eigenvalues are linearly independent.) Once one has done it once one can work it out from scratch the next time it is needed.

[84] The coefficient $\alpha_2[\chi_2(r_0) - \chi_1(r_0)]$ is not zero.

Let $\psi_1, \dots, \psi_{N+1} : R \to L$ be ring homomorphisms. Consider the $N \times (N+1)$ matrix with entries in L

$$M = \begin{bmatrix} \psi_1(u_1) & \dots & \psi_{N+1}(u_1) \\ \vdots & & \vdots \\ \psi_1(u_N) & \dots & \psi_{N+1}(u_N) \end{bmatrix}.$$

The rank-nullity Theorem[85] guarantees that there is a non-zero vector

$$\begin{bmatrix} a_1 \\ \vdots \\ a_{N+1} \end{bmatrix}$$

in the kernel of M, with each $a_i \in L$. Observe that $\psi = \sum_{i=1}^{N+1} a_i\psi_i$ is the zero element of $\mathscr{F}$. Indeed, we have constructed ψ so that $\psi(u_j) = 0$ for all j. Furthermore, if r is an arbitrary element of R, then $r = \sum_{j=1}^{N} n_j u_j$ for some $n_j \in \mathbb{Z}$. Each ψ_i is a ring homomorphism – hence each ψ is a homomorphism of Abelian groups. Thus,

$$\psi_i\left(\sum_{j=1}^{N} n_j u_j\right) = \sum_{j=1}^{N} n_j \psi_i(u_j).$$

The vector space L is also an Abelian group; so

$$\begin{aligned}\psi(r) = \psi\left(\sum_{j=1}^{N} n_j u_j\right) &= \sum_{i=1}^{N+1} a_i\psi_i\left(\sum_{j=1}^{N} n_j u_j\right) = \sum_{i=1}^{N+1} a_i \sum_{j=1}^{N} n_j\psi_i(u_j) \\ &=^* \sum_{j=1}^{N} n_j \sum_{i=1}^{N+1} a_i\psi_i(u_j) = \sum_{j=1}^{N} n_j\psi(u_j) = \sum_{j=1}^{N} n_j(0) = 0.\end{aligned}$$

We marked the interesting equality with $*$. The point is that, if a_i and $\psi_i(u_j)$ are in L, then

$$a_i\Big(\underbrace{\psi_i(u_j) + \cdots + \psi_i(u_j)}_{n_j \text{ times}}\Big) = \underbrace{a_i\psi_i(u_j) + \cdots + a_i\psi_i(u_j)}_{n_j \text{ times}}.$$

Thus, $\sum_{i=1}^{N+1} a_i\psi_i = 0$ with some a_i not zero and $\psi_1, \dots, \psi_{N+1}$ are linearly dependent elements of $\mathscr{F}$.

The wrap up. In step 1, we proved that

$$\psi\circ\sigma \text{ with } \sigma \in \operatorname{Aut}_{\mathbb{Q}} K$$

is a list of N distinct ring homomorphisms from R to L. In steps 2 and 3, we proved that there are at most N distinct ring homomorphisms from R to L. We conclude that if $\psi' : R \to L$ is a ring homomorphism, then $\psi' = \psi\circ\sigma$ for some $\sigma \in \operatorname{Aut}_{\mathbb{Q}} K$.

This completes the proof of (c) and hence the proof of Theorem 5.95. □

[85] The matrix M gives a L-module homomorphism from L^{N+1} to L^N. The theorem asserts that rank M + nullity $M = N + 1$, because $N + 1$ is the dimension of the domain of M. Observe that the rank of M is at most N which is the dimension of the target. Conclude that the nullity of M is at least one.

CONTENTS

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