MATH 702 – SPRING 2024
EXAM 2 SOLUTION

Please e-mail your typed answer to me before class on Monday, April 1.
Let $\zeta = e^{2\pi i/17}$. Exhibit complex numbers $u_1, \dots, u_n$, for some n, such that

$$u_i^2 \in \mathbb{Q}[u_1, \dots, u_{i-1}],$$

for $1 \le i \le n$, and $\zeta \in \mathbb{Q}[u_1, \dots, u_n]$. Please prove your assertions.

Remarks. 1. Please do this on your own as much as you can. If you have to consult things that are not already in your head, try to keep your consultation to be as small as possible.
2. I hope for a Galois Theory argument (rather than a whole bunch of Trigonometry formulas that neither of us know).
3. Of course, by answering this question you are giving a procedure for constructing a regular 17-gon by ruler and compass.

Let $K = \mathbb{Q}[\zeta]$. The minimal polynomial of ζ over $\mathbb{Q}$ is $f(x) = \sum\limits_{j=0}^{16} x^j$. (This polynomial is irreducible by Eisenstein's criterion.) Every root of $f(x)$ is in K; so, K is the splitting field of a separable polynomial over $\mathbb{Q}$; hence, $\mathbb{Q} \subseteq K$ is a Galois extension. Let σ in $\operatorname{Aut}_{\mathbb{Q}} K$ be defined by $\sigma(\zeta) = \zeta^3$. (We know that σ is a well-defined field map because ζ and ζ^3 have the same minimal polynomial over $\mathbb{Q}$.) It is easy to see that the subgroup of $\operatorname{Aut}_{\mathbb{Q}} K$ which is generated by σ has order 16. On the other hand, we know that $|\operatorname{Aut}_{\mathbb{Q}} K| = \dim_{\mathbb{Q}} K = 16$. We conclude that $\operatorname{Aut}_{\mathbb{Q}} K$ is the cyclic group generated by σ. The subgroups of $\operatorname{Aut}_{\mathbb{Q}} K$ are

$$<\sigma> \supseteq <\sigma^2> \supseteq <\sigma^4> \supseteq <\sigma^8> \supseteq <1>.$$

The corresponding fields are:

$$\mathbb{Q} = K^{<\sigma>} \subseteq K^{<\sigma^2>} \subseteq K^{<\sigma^4>} \subseteq K^{<\sigma^8>} \subseteq K = K^{<1>}.$$

Each of the indicated field extensions has dimension two. For each i, with $0 \le i \le 4$, let

$$K_i = K^{\langle \sigma^{2^i} \rangle}.$$

We will identify β_i with

$$K_{i-1}[\beta_i] = K_i,$$

for $1 \le i \le 4$. Then we apply (5) and (6) from Low Lying Fruit to see that if

$$\boxed{u_i = \beta_i - \sigma(\beta_i)},$$

then

$$K_{i-1}[u_i] = K_i, \quad \text{with} \quad u_i^2 \in K_{i-1},$$

for each i with $1 \le i \le 4$.

Define

$$\begin{cases} \beta_1 = \zeta + \zeta^9 + \zeta^{13} + \zeta^{15} + \zeta^{16} + \zeta^8 + \zeta^4 + \zeta^2 \\ \beta_2 = \zeta + \zeta^{13} + \zeta^{16} + \zeta^4 \\ \beta_3 = \zeta + \zeta^{16} \\ \beta_4 = \zeta. \end{cases}$$

Observe that

$$\sigma^{2^i}(\beta_i) = \beta_i \quad \text{and} \quad \sigma^{2^{i-1}}(\beta_i) \neq \beta_i,$$

for each[1] i. Thus, $K_{i-1}[u_i] = K_i$.

Here is an alternate answer, which is also correct: Take

$$\begin{cases} w_1 = \beta_1 - \sigma(\beta_1) \\ w_2 = \beta_2 - \sigma^2(\beta_2) \\ w_3 = \beta_3 - \sigma^4(\beta_3) \\ w_4 = \beta_4 - \sigma^8(\beta_4) \end{cases}$$

Notice that $w_i \neq u_i$, for $2 \le i$; thus, one can not appeal to problem 6 in Low Lying fruit. However, it is clear that $\sigma^2(w_2) = -w_2$. It follows that $w_2 \in K^{\langle \sigma^4 \rangle} \setminus K^{\langle \sigma^2 \rangle}$.

Similarly $\sigma^4(w_3) = -w_3$; hence $w_3 \in K^{\langle \sigma^8 \rangle} \setminus K^{\langle \sigma^4 \rangle}$. Also, $\sigma^8(w_4) = -w_4$; hence

$$w_4 \in K^{\langle \sigma^{16} \rangle} \setminus K^{\langle \sigma^8 \rangle}.$$

One obtains $w_i^2 \in \mathbb{Q}[w_1, \dots, w_{i-1}]$, for each i, and $\zeta \in \mathbb{Q}[w_1, \dots, w_4]$

[1] To show that two elements of K are not equal we use the fact that $\{\zeta^i | 0 \le i \le 15\}$ is a basis for K over $\mathbb{Q}$ and $\sum_{i=0}^{16} \zeta^i = 0$.