EXAM 1 MATH 702 SPRING 2024

Write your answers as **legibly** as you can on the blank sheets of paper provided. Write **complete** answers in **complete sentences**. Make sure that your **notation is defined**!

Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

If some problem is incorrect, then give a counterexample and/or supply the missing hypothesis and prove the resulting statement. If some problem is vague, then be sure to explain your interpretation of the problem.

You should KEEP this piece of paper.

Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. There are four problems.

1. (13 points) Let R be a commutative ring and M and N be R-modules. Prove that there exists an R-module P such that the R-modules M and $N \oplus P$ are isomorphic if and only if there exist R-module homomorphisms $\pi : M \to N$ and $i : N \to M$ such that $\pi \circ i$ is equal to the identity map on N.
2. (13 points) Let R be a commutative ring and let M and N be R-modules. Suppose that every R-submodule of M is finitely generated and every R-submodule of N is finitely generated. Prove that every R-submodule of $M \oplus N$ is finitely generated.
3. (12 points) Let M be the following matrix
$$M = \begin{array}{|c|c|c|} \hline J_2(1) & 0 & 0 \\ \hline 0 & J_3(1) & 0 \\ \hline 0 & 0 & J_4(2) \\ \hline \end{array}\,,$$
where $J_n(a)$ is a Jordan block. What is the dimension of the Null Space of $(M - aI)^n$ for all $a \in \mathbb{C}$ and for all positive integers n?
4. (12 points) Let $\boldsymbol{k}$ be a field. Consider the polynomial rings $R = \boldsymbol{k}[x, y, z]$ and $S = \boldsymbol{k}[t]$. Define a ring homomorphism $\phi : R \to S$ with $\phi(x) = t^3$, $\phi(y) = t^4$, $\phi(z) = t^5$, and $\phi(\alpha) = \alpha$ for all $\alpha \in \boldsymbol{k}$.
   (a) What is the kernel of ϕ?
   (b) Prove that your answer to (a) is correct.