EXAM 1 MATH 702 SPRING 2024

Write your answers as **legibly** as you can on the blank sheets of paper provided. Write **complete** answers in **complete sentences**. Make sure that your **notation is defined**!

Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

If some problem is incorrect, then give a counterexample and/or supply the missing hypothesis and prove the resulting statement. If some problem is vague, then be sure to explain your interpretation of the problem.

You should KEEP this piece of paper.
Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. There are 4 problems.

1. (13 points) **Let R be a commutative ring and M and N be R-modules. Prove that there exists an R-module P such that the R-modules M and $N \oplus P$ are isomorphic if and only if there exist R-module homomorphisms $\pi : M \to N$ and $i : N \to M$ such that $\pi \circ i$ is equal to the identity map on N.**

   ($\Rightarrow$) If $\phi : M \to N \oplus P$ is an isomorphism, then define $i : N \to M$ to be the composition

   $$N \xrightarrow{\text{inclusion}} N \oplus P \xrightarrow{\phi^{-1}} M$$

   and define $\pi : M \to N$ to be the composition

   $$M \xrightarrow{\phi} N \oplus P \xrightarrow{\text{projection}} N.$$

   Observe that $\pi \circ i = \mathrm{id}_N$.

   ($\Leftarrow$) It suffices to show that $M = \operatorname{im} i + \ker \pi$ and $\operatorname{im} i \cap \ker \pi = 0$.

   $M = \operatorname{im} i + \ker \pi$: If $m \in M$, then $m = (i \circ \pi)(m) + m - (i \circ \pi)m$, with $(i \circ \pi)(m)$ in the image of i and $m - (i \circ \pi)m$ in $\ker \pi$.

   $\operatorname{im} i \cap \ker \pi = 0$: If $m \in \operatorname{im} i \cap \ker \pi$, then $m = i(n)$ for some n in N and $0 = \pi(m) = \pi(i(n)) = n$. Thus $n = 0$ and $m = i(n)$ is also zero.
2. (13 points) **Let R be a commutative ring and let M and N be R-modules. Suppose that every R-submodule of M is finitely generated and every R-submodule of N is finitely generated. Prove that every R-submodule of $M \oplus N$ is finitely generated.**

Let X be an R-submodule of $M \oplus N$. The set

$$Y = \left\{ m \in M \,\middle|\, \exists n \in N \text{ with } \begin{bmatrix} m \\ n \end{bmatrix} \in X \right\}$$

is a submodule of M. Every submodule of M is finitely generated. Select $x_1, \dots, x_r \in X$ such that

$$x_i = \begin{bmatrix} m_i \\ n_i \end{bmatrix}, \text{ with } m_i \in M, \text{ and } n_i \in N$$

and $m_1, \dots, m_r$ generate Y. The set

$$Z = \left\{ n \in N \,\middle|\, \begin{bmatrix} 0 \\ n \end{bmatrix} \in X \right\}$$

is a submodule of N. Every submodule of N is finitely generated; hence there exist $x'_1, \dots, x'_s$ in X with $x'_i = \begin{bmatrix} 0 \\ n_i \end{bmatrix}$ and $n_1, \dots, n_s$ generate Z.

It follows that $x_1, \dots, x_r, x'_1, \dots, x'_s$ generate X.

3. (12 points) **Let M be the following matrix**

$$M = \begin{array}{|c|c|c|} \hline J_2(1) & 0 & 0 \\ \hline 0 & J_3(1) & 0 \\ \hline 0 & 0 & J_4(2) \\ \hline \end{array},$$

where $J_n(a)$ is a Jordan block. What is the dimension of the Null Space of $(M - aI)^n$ for all $a \in \mathbb{C}$ and for all positive integers n?

a	n	the dimension of the Null Space of $(M - aI)^n$
1	1	2
1	2	4
1	$3 \leq n$	5
2	1	1
2	2	2
2	3	3
2	$4 \leq n$	4
$a \neq 1, 2$	$1 \leq n$	0

4. (12 points) **Let $\boldsymbol{k}$ be a field. Consider the polynomial rings $R = \boldsymbol{k}[x, y, z]$ and $S = \boldsymbol{k}[t]$. Define a ring homomorphism $\phi : R \to S$ with $\phi(x) = t^3$, $\phi(y) = t^4$, $\phi(z) = t^5$, and $\phi(\alpha) = \alpha$ for all $\alpha \in \boldsymbol{k}$.**

(a) **What is the kernel of ϕ?**

The kernel of ϕ is the ideal $(xz - y^2, x^3 - yz, z^2 - x^2y)$ of R.

(b) **Prove that your answer to (a) is correct.**

The inclusion $(xz - y^2, x^3 - yz, z^2 - x^2y) \subseteq \ker \phi$ is obvious.

Let $\bar{R}$ be the ring

$$\bar{R} = \frac{\boldsymbol{k}[x, y, z]}{(xz - y^2, x^3 - yz, z^2 - x^2y)}.$$

We show $\ker\phi \subseteq (xz - y^2, x^3 - yz, z^2 - x^2y)$ by showing that $\bar{\phi} : \bar{R} \to S$ is injective. Notice that $\{\bar{x}^i, \bar{x}^i\bar{y}, \bar{x}^i\bar{z} \mid 0 \le i\}$ is a basis for $\bar{R}$ as a vector space over $\boldsymbol{k}$.

The proposed basis spans $\bar{R}$ because $\bar{y}^2 = \bar{x}\bar{z}$, $\bar{y}\bar{z} = \bar{x}^3$, and $\bar{z}^2 = \bar{x}^2\bar{y}$. One can always lower the degree in y and z all the way down to at most one 1.

The proposed basis is linearly independent in $\bar{R}$ because ϕ carries the proposed basis to the basis $\{1\} \cup \{t^i \mid 3 \le i\}$ for the image of ϕ.