MATH 701 – FALL 2023
HOMEWORK 7
DUE MONDAY, NOVEMBER 13 BY THE BEGINNING OF CLASS.

17. Prove that A_4 does not have a subgroup of order 6.
18. Let $\phi : \mathbb{Z}/4\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/3\mathbb{Z})$ be the homomorphism with

$$\phi(\bar{b})|_{\bar{c}} = (-1)^b \bar{c}$$

for all $\bar{b} \in \mathbb{Z}/4\mathbb{Z}$ and $\bar{c} \in \mathbb{Z}/3\mathbb{Z}$. (We say this a little more slowly: ϕ is a homomorphism from $\mathbb{Z}/4\mathbb{Z}$ to $\mathrm{Aut}(\mathbb{Z}/3\mathbb{Z})$. If $\bar{b}$ is in $\mathbb{Z}/4\mathbb{Z}$, then $\phi(\bar{b})$ is an automorphism of $\mathbb{Z}/3\mathbb{Z}$. If $\bar{b}$ is in $\mathbb{Z}/4\mathbb{Z}$ and $\bar{c} \in \mathbb{Z}/3\mathbb{Z}$, then $\phi(\bar{b})$ sends $\bar{c}$ to $(-1)^b\bar{c}$.)[1] Let T be the group $\mathbb{Z}/3\mathbb{Z} \rtimes_\phi \mathbb{Z}/4\mathbb{Z}$.

   (a) What is the order of each element of T?

   (b) Identify elements x and y in T with $T = \langle x, y \rangle\ x^6 = \mathrm{id}$, $y^2 = x^3$, and $yxy^{-1} = x^5$.

   (c) Let F be the free group on the two letters X and Y; and let N be the smallest normal subgroup of F which contains X^6, Y^2X^3, $YXY^{-1}X$. Prove that F/N is isomorphic to T.
19. Let $\phi : \mathbb{Z}^4 \to \mathbb{Z}^3$ be the group homomorphism with $\phi(v) = Mv$ for all $v \in \mathbb{Z}^4$, where

$$M = \begin{bmatrix} 3 & 5 & 5 & 6 \\ 2 & 7 & 10 & 7 \\ 3 & 8 & 11 & 9 \end{bmatrix}$$

and Mv is matrix multiplication. Let G be the Abelian group $\mathbb{Z}^3/\mathrm{im}(\phi)$. Every element in G has the form $\bar{w}$, where $w \in \mathbb{Z}^3$.

   (a) Identify elements $w_1, \dots, w_r$ in $\mathbb{Z}^3$, for some r, with $G = \mathbb{Z}\bar{w}_1 \oplus \mathbb{Z}\bar{w}_2 \oplus \cdots \oplus \mathbb{Z}\bar{w}_r$.

   (b) What is the order of the cyclic group $\mathbb{Z}\bar{w}_i$ for each i?
20. Classify the non-Abelian groups of order eight. (This instruction means state and prove a result which says, "If G is a non-Abelian group of order 8, then G is isomorphic to exactly one of the following groups: … .")

[1] If n and a are integers, we write $\bar{a}$ for the class of a in $\mathbb{Z}/n\mathbb{Z}$.