MATH 701 – FALL 2023
HOMEWORK 6
DUE MONDAY, NOVEMBER 6 BY THE BEGINNING OF CLASS.

13. **Let G be a group of order p^n for some prime p and let H be a normal subgroup of G, with $H \neq \{\text{id}\}$. Prove that $Z(G) \cap H \neq \{\text{id}\}$, where $Z(G)$ is the center of G.**

Let G act on H by conjugation. It follows that

$$|H| = |\{h \in H \mid ghg^{-1} = h \text{ for all } g \in G\}| + \sum_x [G : \text{stab}(x)],$$

where the sum is taken over all orbits of size more than 1 and we take one element x from each orbit. Observe that

$$\{h \in H \mid ghg^{-1} = h \text{ for all } g \in G\} = Z(G) \cap H.$$

We see that p divides the order of H and p divides each summand $[G : \text{stab}(x)]$. So p divides $|Z(G) \cap H|$. Of course, the identity element is in $Z(G) \cap H$; so this subgroup must have order at least p.

14. **How many elements of order 7 are there in a simple group of order 168?**

Let G be a simple group with $168 = 2^3 \cdot 3 \cdot 7$ elements. The Sylow Theorems tell us that the number of 7-element subgroups of G is congruent to 1 mod 7 and divides 168. This number must be greater than 1 because G has no normal subgroups. Thus, there are exactly 8 seven-element subgroups of G. Each non-identity element of such a group generates the entire subgroup; thus, any pair of such subgroups intersect to only $\{\text{id}\}$. Each such subgroup has 6 elements of order 7. Thus, G has exactly 48 elements of order 7.

15. **Classify all groups of order $2p$ where p is an odd prime integer. (This instruction means state and prove a result which says, "If G is a group of order $2p$, where p is an odd prime integer, then G is isomorphic to exactly one of the following groups: $\dots$.")**

Theorem. *If G is a group of order $2p$, where p is an odd prime integer, then G is a cyclic group or a dihedral group.*

Proof. Let G be a group of order $2p$. The Sylow Theorems guarantee that G has an element b of order p and an element a of order 2. The subgroup $\langle b \rangle$ has index 2 in G; hence, $\langle b \rangle$ is a normal subgroup of G. We also know that $G = \langle a, b \rangle$.

We first suppose that the subgroup $\langle a \rangle$ of G is also normal in G. It follows that gag^{-1} is an element of order 2 in $\langle a \rangle$ for all $g \in G$; and therefore, $gag^{-1} = a$ for all $g \in G$, and G is abelian. There is no difficulty in showing that G is generated by ab.

Henceforth, we assume that $\langle a \rangle$ is not a normal subgroup of G. The Sylow Theorems guarantee that G has more than one sugroup of order 2 and the number of subgroups of order 2 is odd and divides $2p$. Thus, G must have p subgroups of order 2. In other words, every element of G

that is not in $\langle b\rangle$ has order 2. It follows that ab has order two. Thus, G is a group of order $2p$ with G generated by a and b with $a^2 = (ab)^2 = b^p = \text{id}$. It follows that G is a dihedral group. □

16. **Let G be a group of order 30. Prove that G has a subgroup of order 15.**

Apply the Sylow Theorems. There is a subgroup H or order 3 and a subgroup K of order 5. If either H or K is a normal subgroup of G, then HK is a subgroup of G of order 15.

Let n_3 be the number of Sylow 3-subgroups of G and n_5 be the number of Sylow 5-subgroups of G. As observed above, it suffices to prove that $n_3 = 1$ or $n_5 = 1$. Assume $n_3 \neq 1$ and $n_5 \neq 1$. We hope to reach a contradiction. Apply the Sylow Theorems again. The number n_3 is congruent to 1 mod 3 and n_3 divides 10. The only remaining option is $n_3 = 10$. The number n_5 is congruent to one mod 5 and n_5 divides 6. The only option left is $n_5 = 6$. Every element of order three is in exactly one Sylow 3-subgroup. Thus there are $2 \times 10 = 20$ elements of order 3. Every element of order 5 is in exactly one Sylow 5-subgroup. Thus, there are $4 \times 6 = 24$ elements of order 5. We have already reached a contradiction because $30 < 20 + 24$. □