I changed my mind. I decided that it is not reasonable to make you turn in these problems before the exam.

MATH 701 – FALL 2023
HOMEWORK 5
DUE MONDAY, OCTOBER 30 BY THE BEGINNING OF CLASS.

9. **(True or False) If true, prove it. If false, give a counterexample. Let G be a group. If x and y are elements of G of finite order, then xy has finite order.**

   The assertion is False. I offer three examples.

   Example 1. Let $\mathscr{G}$ be the group of rigid motions of the plane with operation composition. Let σ be reflection across the x-axis and ρ be rotation fixing the origin by 1-radian. Notice that σ has order two and ρ has infinite order. Notice also that $\sigma\rho$ is a reflection; so $\sigma\rho$ has order two. Thus, σ and $\sigma\rho$ each have order two, but $\sigma(\sigma\rho) = \rho$ has infinite order.

   Example 2. Let $\mathscr{G}$ be the group of rigid motions of the plane with operation composition; let ℓ_1 and ℓ_2 be parallel lines in the plane; and let σ_i be reflection across ℓ_i. Observe that σ_1 and σ_2 have order two; but $\sigma_1\sigma_2$ is translation which has infinite order. (An undergraduate student gave me this solution a number of years ago.)

   Example 3. Let G be the group $\mathrm{GL}(2,\mathbb{R})$, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$. Calculate A has order 2; B has order 4, but $AB = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ has infinite order. Indeed, $(AB)^n = \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}$, where the f's are the Fibonacci numbers: $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_n + f_{n+1}$ for $0 \le n$. (Many undergraduate students gave me this solution in the last year or so. I assume the solution is on the Internet somewhere.)

10. **(True or False) If true, prove it. If false, give a counterexample. Let G be a group. If x and y are elements of G of finite order and $xy = yx$, then the order of xy is equal to the least common multiple of the order of x and the order of y.**

    The assertion is false. Let x be an n-cycle and $y = x^{-1}$. So, y is also an n-cycle. It follows that x and y each have order n, but xy has order 1. Of course, 1 is not equal to the least common multiple of n and n, unless n happens to be 1.

11. **Let I be an index set. Suppose that for each $i \in I$, G_i is a group. Consider the direct product $\prod_{i\in I} G_i$. For each $i_0 \in I$, let $\mathrm{proj}_{i_0} : \prod_{i\in I} G_i \to G_{i_0}$ be the natural projection map. Let G be a group and, for each i, let $\phi_i : G \to G_i$ be a group homomorphism. Prove that there exists a unique group homomorphism $\Phi : G \to \prod_{i\in I} G_i$ so that the diagram**

(0.0.1)
$$\begin{array}{ccc} G & \overset{\exists!\Phi}{\dashrightarrow} & \prod_{i\in I} G_i \\ & \searrow^{\phi_{i_0}} & \downarrow{\scriptstyle \mathrm{proj}_{i_0}} \\ & & G_{i_0} \end{array}$$

commutes for all $i_0 \in I$.

We first address the uniqueness issue. Suppose $\Phi : G \to \prod_{i\in I} G_i$ is a homomorphism which causes (0.0.1) to commute. Let $g \in G$. Then $\Phi(g)$ is an element in $\prod_{i\in I} G_i$; thus, $\Phi(g)$ is equal to $(h_i)_{i\in I}$, for some $h_i \in G_i$, for all $i \in I$. The hypothesis that the commutative diagram (0.0.1) commutes ensures that

$$\phi_{i_0}(g) = \mathrm{proj}_{i_0}(\Phi(g)) = \mathrm{proj}_{i_0}\left((h_i)_{i\in I}\right) = h_{i_0},$$

for each $i_0 \in H$. In other words, if Φ exists and (0.0.1) commutes, then $\Phi(g)$ must equal $(\phi_i(g))_{i\in I}$ for all $g \in G$.

Now we prove the existence of a homomorphism $\Phi : G \to \prod_{i\in I} G_i$ which satisfies (0.0.1). If $g \in G$, then $(\phi_i(g))_{i\in I}$ is a legitimate element of $\prod_{i\in I} G_i$. We define

$$\Phi(g) = (\phi_i(g))_{i\in I}.$$

We show that Φ is a group homomorphism. Take g, h in G. Observe that

$$\Phi(gh) = (\phi_i(gh))_{i\in I} = (\phi_i(g)\phi_i(h))_{i\in I} = (\phi_i(g))_{i\in I} \cdot (\phi_i(h))_{i\in I} = \Phi(g) \cdot \Phi(h).$$

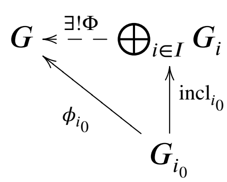
We show that Φ causes (0.0.1) to commute. Take $g \in G$. We see that

$$(\mathrm{proj}_{i_0} \circ \Phi)(g) = \mathrm{proj}_{i_0}((\phi_i(g))_{i\in I}) = \phi_{i_0}(g),$$

as desired.

12. **Let I be an index set. Suppose that for each $i \in I$, G_i is a group. Consider the direct sum $\bigoplus_{i\in I} G_i$. For each $i_0 \in I$, let $\mathrm{incl}_{i_0} : G_{i_0} \to \bigoplus_{i\in I} G_i$ be the natural inclusion map. Let G be an Abelian group and, for each i, let $\phi_i : G_i \to G$ be a group homomorphism. Prove that there exists a unique group homomorphism $\Phi : \bigoplus_{i\in I} G_i \to G$ so that the diagram**

(0.0.2)



commutes for all $i_0 \in I$.

The group G is Abelian. Let the operation of G be called $+$ and the identity element of G be called 0. If L is a finite list of elements of G, then $\sum_{\ell \text{ in the list } L} \ell$ is a legitimate element of G. We write the operation of G_i as $*_i$ and the operation of $\bigoplus_{i\in I} G_i$ as $\times$. In particular,

$$(g_i)_{i\in I} \times (h_i)_{i\in I} = (g_i *_{G_i} h_i)_{i\in I},$$

for $(g_i)_{i\in I}$ and $(h_i)_{i\in I}$ in $\bigoplus_{i\in I} G_i$.

The proof has two parts.

(A) We prove that if $\Phi : \bigoplus_{i\in I} G_i \to G$ is a group homomorphism for which (0.0.2) commutes and $(g_i)_{i\in I}$ is in $\bigoplus_{i\in I} G_i$, then

$$\Phi((g_i)_{i\in I}) = \sum_{\{i\in I \mid g_i \neq \mathrm{id}\}} \phi_i(g_i). \tag{0.0.3}$$

(B) We prove that the formula of (0.0.3) describes a well-defined function $\Phi : \bigoplus_{i\in I} G_i \to G$; furthermore, this function is a group homomorphism and (0.0.2) commutes.

(12A) Let $\theta = (g_i)_{i\in I}$ be an arbitrary element of $\bigoplus_{i\in I} G_i$. Let $I' = \{i \in I \mid g_i \neq \mathrm{id}\}$. The definition of direct sum ensures that I' is finite. Observe that $\theta = \prod_{i\in I'} \mathrm{incl}_i(g_i)$. The hypothesis that Φ is a group homomorphism for which (0.0.2) commutes ensures that

$$\Phi(\theta) = \Phi\left(\prod_{i\in I'} \mathrm{incl}_i(g_i)\right) = \sum_{i\in I'} \Phi\Big(\mathrm{incl}_i(g_i)\Big) = \sum_{i\in I'} \phi_i(g_i).$$

Thus, there is at most one homomorphism $\Phi : \bigoplus_{i\in I} G_i \to G$ for which (0.0.2) commutes and that homomorphism (if it exists) must satisfy (0.0.3).

(12B) We observed above that if $(g_i)_{i\in I}$ is an arbitrary element of $\bigoplus_{i\in I} G_i$, then $\sum\limits_{\{i\in I \mid g_i\neq \mathrm{id}\}} \phi_i(g_i)$ is a well-defined element of G. It follows that (0.0.3) describes a well-defined function

$$\Phi : \bigoplus_{i\in I} G_i \to G.$$

We prove that Φ is a homomorphism. Let $(g_i)_{i\in I}$ and $(h_i)_{i\in I}$ be elements of $\bigoplus_{i\in I} G_i$. Observe that

$$\begin{aligned}
\Phi\Big((g_i)_{i\in I} \times (h_i)_{i\in I}\Big) &= \Phi\Big((g_i *_{G_i} h_i)_{i\in I}\Big) \\
&= \sum_{\{i\in I \mid g_i *_{G_i} h_i \neq \mathrm{id}\}} \phi_i(g_i *_{G_i} h_i) \\
&= \sum_{\{i\in I \mid g_i *_{G_i} h_i \neq \mathrm{id}\}} \Big(\phi_i(g_i) + \phi_i(h_i)\Big) \\
&= \sum_{\{i\in I \mid g_i \neq \mathrm{id}\}} \phi_i(g_i) + \sum_{\{i\in I \mid h_i \neq \mathrm{id}\}} \phi_i(h_i) \qquad \text{One must think about this step.} \\
&= \Phi\Big((g_i)_{i\in I}\Big) + \Phi\Big((h_i)_{i\in I}\Big)
\end{aligned}$$

We prove that (0.0.2) commutes. Fix i_0 and let g_{i_0} be an element of G_{i_0}. Observe that

$$\begin{aligned}
(\Phi \circ \mathrm{incl}_{i_0})(g_{i_0}) &= \Phi\Big((h_i)_{i\in I}\Big), \qquad \text{where } h_i = \begin{cases} \mathrm{id}_{G_i} & \text{if } i \neq i_0 \\ g_{i_0} & \text{if } i = i_0 \end{cases} \\
&= \sum_{\{i\in I \mid h_i \neq \mathrm{id}\}} \phi_i(h_i) = \phi_{i_0}(g_{i_0}),
\end{aligned}$$

as desired.