MATH 701 – FALL 2023
HOMEWORK 4
DUE MONDAY, OCTOBER 9 BY THE BEGINNING OF CLASS.

8. (Recall that index of the subgroup H of the group G is the number of left cosets of H in G and this number is denoted $|G : H|$.) **Let G be a group and $H \subseteq K$ be subgroups of G of finite index. What formula relates the three numbers $|G : H|$, $|G : K|$, and $|K : H|$? Prove your formula. Notice that H and K are not assumed to be normal subgroups of G, and they are also not assumed to be finite.**

Observation. *Let G be a group and $H \subseteq K$ be subgroups of G of finite index. Then*

$$|G : H| = |G : K||K : H|.$$

Proof. Write G/H, G/K, and K/H for the set of left cosets of H in G, the set of left cosets of K in G, and the set of left cosets of H in K, respectively. Let $N = |K : H|$ and select $k_1, \dots, k_N$ in K with $K/H = \{k_1H, \dots, k_NH\}$

Observe that $\Phi : G/H \to G/K$, given by $\Phi(gH) = gK$, for all $g \in H$, is a (well-defined) surjective function. Indeed, if g and g' are in G, with $gH = g'H$, then $g = g'h$ for some $h \in H$, and

$$gK = g'hK = g'K.$$

(The last equality holds because $H \subseteq K$.) Thus, Φ is a (well-defined) function. It is obvious that Φ is surjective. (Indeed, if gK is an arbitrary element of G/K, then $gH$is an element of G/H and $\Phi(gH) = gK$.)

Claim. *Let gK be an arbitrary element of G/K. We claim that there are exactly N elements of G/H which are sent to gK under Φ.*

Proof of Claim. Observe that for each i with $1 \le i \le N$,

$$\Phi(gk_iH) = gk_iK = gK.$$

Observe that the elements

$$gk_1H, \dots, gk_NH$$

of G/H are distinct. Observe that if $\Phi(g'H) = gK$, for some g' in G, then $g'K = gK$ and $g^{-1}g' \in K$. But K is equal to the disjoint union $\bigcup_{i=1}^{N} k_iH$. Thus, $g^{-1}g' \in k_iH$, for some i, and $g^{-1}g' = k_ih$ for some $h \in H$. It follows that $g'H = gk_iH$. This completes the proof of the claim. □

This also completes the proof of the Observation. We have partitioned G/H into the disjoint union of $|G : K|$ sets and each of these sets has $|K : H|$ elements. □