MATH 701 – FALL 2023
HOMEWORK 3
DUE MONDAY, OCTOBER 2 BY THE BEGINNING OF CLASS.

7. **Use the ideas in the proof of Cayley's Theorem to find permutations** $a, b \in S_8$ **which satisfy:** $a^4 = 1$**,** $b^2 = a^2$**,** $ba = a^3b$**, and the permutations** a^ib^j**, with** $0 \le i \le 3$ **and** $0 \le j \le 1$**, are distinct.**

   We hope to produce a group G whose elements are $1, \alpha, \alpha^2, \alpha^3, \beta, \alpha\beta, \alpha^2\beta, \alpha^3\beta$ and which satisfy the relations $\alpha^4 = 1$, $\beta^2 = \alpha^2$, $\beta\alpha = \alpha^3\beta$. We number these elements as element number 1 up to element number 8, in the listed order. We will act like this group exists. We will apply the idea of Cayley's to find a subgroup of S_8 which is isomorphic to the desired group G; hence we will have a concrete version of G in our hands and we will be able to see that our concrete version has 8 distinct elements. Anyhow left multiplication by α sends

   $$1 \to \alpha \to \alpha^2 \to \alpha^3 \to 1 \quad \text{and} \quad \beta \to \alpha\beta \to \alpha^2\beta \to \alpha^3\beta \to \beta.$$

   So, left multiplication by α corresponds to the permuation $a = (1,2,3,4)(5,6,7,8)$ in S_8. Also, left multiplication by β sends

   $$1 \to \beta \to \alpha^2 \to \alpha^2\beta \to 1 \quad \text{and} \quad \alpha \to \alpha^3\beta \to \alpha^3 \to \alpha\beta \to \alpha.$$

   So, left multiplication by β corresponds to the permuation $b = (1,5,3,7)(2,8,4,6)$ in S_8. It is now easy to see that

   $$\begin{gathered} (1) \\ a = (1,2,3,4)(5,6,7,8) \\ a^2 = (1,3)(2,4)(5,7)(6,8) \\ a^3 = (1,4,3,2)(5,8,7,6) \\ b = (1,5,3,7)(2,8,4,6) \\ ab = (1,6,3,8)(2,5,4,7) \\ a^2b = (1,7,3,5)(2,6,4,8) \\ a^3b = (1,8,3,6)(2,7,4,5) \end{gathered}$$

   are distinct and satisfy $a^4 = 1$, $b^2 = a^2$, $ba = a^3b$.