# MATH 701 – FALL 2019
# HOMEWORK 2
# DUE MONDAY, SEPTEMBER 16 BY THE BEGINNING OF CLASS.
# SOLUTIONS

3. **Let $n$ be a fixed positive integer, and let $\mathbb{C}^*$ be the group $\mathbb{C} \setminus \{0\}$ under multiplication. How many subgroups of $\mathbb{C}^*$ have $n$ elements? What are they? Justify your answer.**

$\boxed{\text{There is one subgroup of } \mathbb{C}^* \text{ with } n \text{ elements and it is } \{e^{2\pi i k/n} \mid 0 \le k \le n-1\}.}$

*Proof.* Let $H$ be a subgroup of $\mathbb{C}^*$ with $n$ elements. Observe first that every element of $H$ has modulus 1. Indeed, if $z$ does not have modulus 1, then $z^m$ does not have modulus 1 for any positive integer $m$ and so any subgroup of $\mathbb{C} \setminus \{0\}$ which contains $z$ is necessarily infinite. Let $\theta > 0$ be the smallest positive real number with $e^{i\theta} \in H$. I first claim that if $\alpha$ is a real number with $e^{i\alpha} \in H$, then $\alpha = k\theta$ for some integer $k$. Indeed, write $\alpha = k\theta + r$ for some integer $k$ and some real number $r$ with $0 \le r < \theta$. Observe that $e^{ir} = e^{i\alpha}\left(e^{i\theta}\right)^{-k}$ is in $H$. Our choice of $\theta$ ensures that $r = 0$. The first claim is established. At this point we have shown that $H = \{e^{ik\theta} \mid k \in \mathbb{Z}\}$. I next claim that $2\pi = m\theta$ for some integer $m$. This follows from the first claim because $1 = e^{2\pi i} \in H$. At this point we know that $H = \{e^{2\pi i k/m} \mid k \in \mathbb{Z}\}$. The left side has $n$ elements The right side has $m$ elements. It follows that $n = m$ and the argument is complete. □

4. **Let $C_2$ be the subgroup $\{1, -1\}$ of $\mathbb{C}^*$. Consider the group $G = C_2 \times C_2 \times C_2 \times C_2$, which is the direct product of four copies of $C_2$. How many four element subgroups does $G$ have? Justify your answer.**

$\boxed{\text{There are 35 four-element subgroups of } G.}$

*Proof.* The group $G$ is Abelian. Each element of $G$ squares to the identity element. Each choice of two distinct elements $g_1, g_2$ from $G \setminus \{(1,1,1,1)\}$ gives rise to the four element subgroup

$$\{g_1,\ g_2,\ g_1g_2,\ (1,1,1,1)\} \tag{0.0.1}$$

of $G$. Of course, each subgroup of $G$ has been counted three times in (0.0.1). The group $G$ has 16 elements. The set $G \setminus \{(1,1,1,1)\}$ has 15 elements. There are $\binom{15}{2}$ ways to select a two element subset from a fifteen element set. We have counted each 4-element subgroup of $G$ three times. Thus the number of four element subgroups of $G$ is

$$\frac{1}{3}\binom{15}{2} = \frac{15 \cdot 14}{3 \cdot 2} = 5 \cdot 7 = 35.$$

□

5. **Prove that every element of $\mathrm{SO}_n(\mathbb{R})$ is diagonalizable over $\mathbb{C}$. (It might make sense to prove a more general statement. An element $M$ of $\mathrm{GL}_n(\mathbb{C})$ is called <u>unitary</u> if $\overline{M}^{\mathrm{T}} M$ is the identity**

**matrix. Prove that every unitary matrix from $\mathrm{GL}_n(\mathbb{C})$ is diagonalizable. In this problem $\bar{\ }$ means complex conjugate. Recall that $\mathrm{SO}_n(\mathbb{R})$ is the subgroup of $\mathrm{GL}_n(\mathbb{R})$ which consists of all matrices $M$ with $M^{\mathrm{T}}M$ equal to the identity matrix.)**

**Theorem.** *Every unitary matrix from $\mathrm{GL}_n(\mathbb{C})$ is diagonalizable.*

The proof is a consequence of the following Claim; just take $V$ to be all of $\mathbb{C}^n$.

**Claim.** *Let $M$ be a unitary $n \times n$ matrix with complex entries and let $V$ be a subspace of $\mathbb{C}^n$ with the property that $MV \subseteq V$. Then the restriction of $M$ to $V$ is diagonalizable.*

*Proof.* Write $M|_V$ for the "restriction of $M$ to $V$".

We prove the claim by induction on the dimension of $V$. If $\dim V = 1$, and $v$ is a non-zero element of $V$, then $v$ is a basis for $V$. The hypothesis that $MV \subseteq V$ guarantees that $v$ is an eigenvalue of $M$ and hence $M|_V$ is diagonalizable.

Now suppose that $1 < \dim V$. Recall that $M|_V$ has a non-zero eigenvector. [1]

Let $v_0$ be a non-zero eigenvector of $M|_V$ which belongs to the eigenvalue $\lambda_0$. (The matrix $M$ is non-singular, so $\lambda_0 \neq 0$.) Let

$$W = \{w \in V \mid \bar{w}^{\mathrm{T}} v_0 = 0\}.$$

It is clear that $W$ is a vector space. Observe that

- $V = \mathbb{C}v_0 \oplus W$, and
- $MW \subseteq W$.

Once we are confident with these assertions then the proof of the claim is complete by induction because $\dim W < \dim V$. We establish the two assertions.

We first show that $MW \subseteq W$. If $w \in W$, then $\bar{w}^{\mathrm{T}} v_0 = 0$ and

$$(\overline{Mw})^{\mathrm{T}} v_0 = \frac{1}{\lambda_0} (\overline{Mw})^{\mathrm{T}} M v_0 = \frac{1}{\lambda_0} \bar{w}^{\mathrm{T}} \overline{M}^{\mathrm{T}} M v_0 = \frac{1}{\lambda_0} \bar{w}^{\mathrm{T}} \,\mathrm{id}\, v_0 = \bar{w}^{\mathrm{T}} v_0 = 0;$$

hence, $Mw$ is in $W$, as claimed.

Now we show that $V$ is contained in the sum of $W$ and $\mathbb{C}v_0$. If $v$ is an arbitrary element of $V$, then

$$v = (v - \frac{\bar{v}^{\mathrm{T}} v_0}{\bar{v}_0^{\mathrm{T}} v_0} \cdot v_0) + \frac{\bar{v}^{\mathrm{T}} v_0}{\bar{v}_0^{\mathrm{T}} v_0} \cdot v_0$$

with $(v - \frac{\bar{v}^{\mathrm{T}} v_0}{\bar{v}_0^{\mathrm{T}} v_0} \cdot v_0) \in W$ and $\frac{\bar{v}^{\mathrm{T}} v_0}{\bar{v}_0^{\mathrm{T}} v_0} \cdot v_0 \in \mathbb{C}v_0$

Finally, the intersection of $W$ and $\mathbb{C}v_0$ is zero because $\bar{v}_0^{\mathrm{T}} v_0$ is not zero. $\square$

The Claim has been established. Thus, as was noted above the claim, the Theorem is also established.

[1] Indeed, the characteristic polynomial of $M|_V$ is a polynomial in one variable with complex coefficients. Such a polynomial has a root, say $\lambda_1$ in $\mathbb{C}$ (by the "Fundamental Theorem of Algebra"). Thus $M|_V - \lambda_1 \,\mathrm{id}$ is a singular matrix. Any non-zero vector in the null space of $M|_V - \lambda_1 \,\mathrm{id}$ is an eigenvector of $M|_V$.

6. **Let $\ell$ be the line in 3-space through the origin and parallel to the vector $\vec{i}+2\vec{j}+3\vec{k}$. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be rotation by $\pi/4$ radians where $\ell$ is the axis of revolution. Find a matrix $M$ so that**

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = M \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

**There are two correct answers.**

Apply the algorithm given in class. Take $v_3$ to be the unit vector $v_3 = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, which is parallel to $\ell$. Pick $v_1$ and $v_2$ so that $v_1, v_2, v_3$ form an orthonormal set. I take

$$v_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \frac{1}{\sqrt{70}}\begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix}.$$

Let $Q$ be the matrix that carries $e_1$ to $v_1$; $e_2$ to $v_2$; and $e_3$ to $v_3$, where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is the "standard basis" of $\mathbb{R}^3$. We see that

$$Q = \frac{1}{\sqrt{70}}\begin{bmatrix} 2\sqrt{14} & -3 & \sqrt{5} \\ -\sqrt{14} & -6 & 2\sqrt{5} \\ 0 & 5 & 3\sqrt{5} \end{bmatrix}.$$

The matrix $Q^{-1}$, which is the same as $Q^{\mathrm{T}}$, carries $v_i$ to $e_i$, for all $i$. (Be sure to check that $Q^{\mathrm{T}}Q$ really is the identity matrix.) There are two matrices which fix the $z$-axis and rotate the $xy$-plane by $\pi/4$; namely,

$$A = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{2}\begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 \\ -\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

We now compute

$$\begin{aligned} M &= QAQ^{\mathrm{T}} \\ &= \frac{1}{140}\begin{bmatrix} 2\sqrt{14} & -3 & \sqrt{5} \\ -\sqrt{14} & -6 & 2\sqrt{5} \\ 0 & 5 & 3\sqrt{5} \end{bmatrix}\begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2\sqrt{14} & -\sqrt{14} & 0 \\ -3 & -6 & 5 \\ \sqrt{5} & 2\sqrt{5} & 3\sqrt{5} \end{bmatrix} \\ &= \boxed{\frac{1}{28}\begin{bmatrix} 2+13\sqrt{2} & 4-2\sqrt{2}+6\sqrt{7} & 6-3\sqrt{2}-4\sqrt{7} \\ 4-2\sqrt{2}-6\sqrt{7} & 8+10\sqrt{2} & 12-6\sqrt{2}+2\sqrt{7} \\ 6-3\sqrt{2}+4\sqrt{7} & 12-6\sqrt{2}-2\sqrt{7} & 18+5\sqrt{2} \end{bmatrix}}. \end{aligned}$$

In a similar manner we compute

$$QBQ^{\mathrm{T}} = \boxed{\frac{1}{28}\begin{bmatrix} 2+13\sqrt{2} & 4-2\sqrt{2}-6\sqrt{7} & 6-3\sqrt{2}+4\sqrt{7} \\ 4-2\sqrt{2}+6\sqrt{7} & 8+10\sqrt{2} & 12-6\sqrt{2}-2\sqrt{7} \\ 6-3\sqrt{2}-4\sqrt{7} & 12-6\sqrt{2}+2\sqrt{7} & 18+5\sqrt{2} \end{bmatrix}}.$$

It is easy to check that $Mv_3 = v_3$, as expected, for either choice of $M$.