MATH 701 – FALL 2023
HOMEWORK 1
DUE MONDAY, SEPTEMBER 11 BY THE BEGINNING OF CLASS.

1. (a) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function which fixes the origin and rotates the xy-plane by the angle θ radians in the counterclockwise direction. What matrix satisfies

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = M \begin{bmatrix} x \\ y \end{bmatrix}?$$

(b) Let ℓ be the line through the origin making an angle ϕ with the positive x-axis. (Measure ϕ in radians. Measure angles in the counterclockwise direction starting at the positive x-axis.) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function which reflects the xy-plane across ℓ. What matrix satisfies

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = M \begin{bmatrix} x \\ y \end{bmatrix}?$$

(c) Fill in the blanks (and justify your answers):

- (reflection across the line with angle ϕ_1)$\circ$ (reflection across the line with angle ϕ_2) is ____
- (reflection across the line with angle ϕ)$\circ$ (rotation by θ) is ____
- (rotation by θ)$\circ$ (reflection across the line with angle ϕ) is ____

Note.

(i) View the left side of the statements in (c) as statements about functions. Recall that the function $f \circ g$ acts on x by $(f \circ g)(x) = f(g(x))$.

(ii) The right side of the statements in (c) will read either "rotation by _" or "reflection across the line with angle _". Of course, you will fill in the _.

(iii) Once you complete problem (1) you will have shown that

$$\{f : \mathbb{R}^2 \to \mathbb{R}^2 \mid f \text{ is either a rotation fixing the origin or a reflection across the line through the origin}\}$$

is a group. We call this group $\mathscr{G}$.

2. Let S be the square in the xy-plane with vertices: $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, 0)$, and $v_4 = (0, -1)$. Let D_4 be the subgroup of $\mathscr{G}$ (from problem 1) which carries S onto itself. Let σ be reflection across the x-axis, so $\sigma = (2, 4)$; and let ρ be rotation by $\pi/2$ radians counterclockwise, so $\rho = (1, 2, 3, 4)$.

(a) Write reflection across the lines $y = x$, the y-axis, and $y = -x$ as permutations of the vertices and in the form

$$\sigma^i \rho^j \quad \text{for } 0 \le i \le 1 \text{ and } 0 \le j \le 3. \tag{0.0.1}$$

(b) Complete the following multiplication table for D_4. All entries should be of the form (0.0.1).

	1	ρ	ρ^2	ρ^3	σ	$\sigma\rho$	$\sigma\rho^2$	$\sigma\rho^3$
1								
ρ								
ρ^2								
ρ^3								
σ								
$\sigma\rho$								
$\sigma\rho^2$								
$\sigma\rho^3$								