MATH 701 – FALL 2023
HOMEWORK 1
DUE MONDAY, SEPTEMBER 11 BY THE BEGINNING OF CLASS.

1. (a) **Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function which fixes the origin and rotates the xy-plane by the angle θ radians in the counterclockwise direction. What matrix satisfies**
$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = M \begin{bmatrix} x \\ y \end{bmatrix}?$$
Think of the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in polar coordinates. There are real numbers r and ϕ with $x = r\cos\phi$ and $y = r\sin\phi$. It is clear that
$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{bmatrix} = \begin{bmatrix} r(\cos\theta\cos\phi - \sin\theta\sin\phi) \\ r(\cos\theta\sin\phi + \sin\theta\cos\phi) \end{bmatrix} = \begin{bmatrix} \cos\theta\, x - \sin\theta\, y \\ \cos\theta\, y + \sin\theta\, x \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$
Thus,
$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = M \begin{bmatrix} x \\ y \end{bmatrix}$$
for
$$\boxed{M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.}$$

(b) **Let ℓ be the line through the origin making an angle ϕ with the positive x-axis. (Measure ϕ in radians. Measure angles in the counterclockwise direction starting at the positive x-axis.) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function which reflects the xy-plane across ℓ. What matrix satisfies**
$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = M \begin{bmatrix} x \\ y \end{bmatrix}?$$
If $\phi = 0$, then it is clear that $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. I propose that we think of the given function f as the composition of three functions that we already understand:
$$f = (\text{rotation by } \phi) \circ (\text{reflection across the } x\text{-axis}) \circ (\text{rotation by } -\phi).$$
The matrix for f is the product of the matrices for each of the three pieces. That is,
$$f \begin{bmatrix} x \\ y \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix},$$
where
$$M = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\phi - \sin^2\phi & 2\sin\phi\cos\phi \\ 2\sin\phi\cos\phi & -\cos^2\phi + \sin^2\phi \end{bmatrix} = \boxed{\begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}}$$

(c) **Fill in the blanks (and justify your answers):**

- **(reflection across the line with angle ϕ_1)$\circ$ (reflection across the line with angle ϕ_2) is ____** $\boxed{\text{rotation by the angle } 2(\phi_1 - \phi_2).}$

  Proof. We see that

  (reflection across the line with angle ϕ_1) $\circ$ (reflection across the line with angle ϕ_2)

  is given by multiplication by the matrix

$$\begin{bmatrix} \cos 2\phi_1 & \sin 2\phi_1 \\ \sin 2\phi_1 & -\cos 2\phi_1 \end{bmatrix} \begin{bmatrix} \cos 2\phi_2 & \sin 2\phi_2 \\ \sin 2\phi_2 & -\cos 2\phi_2 \end{bmatrix} = \begin{bmatrix} \cos 2(\phi_1 - \phi_2) & -\sin 2(\phi_1 - \phi_2) \\ \sin 2(\phi_1 - \phi_2) & \cos 2(\phi_1 - \phi_2) \end{bmatrix}. \quad \square$$

  □

- **(reflection across the line with angle ϕ)$\circ$ (rotation by θ) is ____**

  $\boxed{\text{reflection across the line with angle } \phi - \frac{\theta}{2}.}$

  Proof. We see that

  (reflection across the line with angle ϕ)$\circ$ (rotation by θ)

  is given by multiplication by the matrix

$$\begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(2\phi - \theta) & \sin(2\phi - \theta) \\ \sin(2\phi - \theta) & -\cos(2\phi - \theta) \end{bmatrix}. \quad \square$$

  □

- **(rotation by θ)$\circ$ (reflection across the line with angle ϕ) is ____**

  $\boxed{\text{reflection across the line with angle } \phi + \frac{\theta}{2}.}$

  Proof. We see that

  (rotation by θ) $\circ$ (reflection across the line with angle ϕ)

  is given by multiplication by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} = \begin{bmatrix} \cos(2\phi + \theta) & \sin(2\phi + \theta) \\ \sin(2\phi + \theta) & -\cos(2\phi + \theta) \end{bmatrix}. \quad \square$$

  □

Note.

(i) **View the left side of the statements in (c) as statements about functions. Recall that the function $f \circ g$ acts on x by $(f \circ g)(x) = f(g(x))$.**

(ii) **The right side of the statements in (c) will read either "rotation by _" or "reflection across the line with angle _". Of course, you will fill in the _.**

(iii) **Once you complete problem (1) you will have shown that**

$$\{f : \mathbb{R}^2 \to \mathbb{R}^2 \mid f \text{ is either a rotation fixing the origin or a reflection across the line through the origin}\}$$

is a group. We call this group $\mathscr{G}$.

2. **Let S be the square in the xy-plane with vertices: $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (-1,0)$, and $v_4 = (0,-1)$. Let D_4 be the subgroup of $\mathscr{G}$ (from problem 1) which carries S onto itself. Let σ be reflection across the x-axis, so $\sigma = (2,4)$; and let ρ be rotation by $\pi/2$ radians counterclockwise, so $\rho = (1,2,3,4)$.**
   (a) **Write reflection across the lines $y = x$, the y-axis, and $y = -x$ as permutations of the vertices and in the form**

$$\sigma^i \rho^j \quad \text{for } 0 \le i \le 1 \text{ and } 0 \le j \le 3. \tag{0.0.1}$$

One straightforward calculation is:

$$\begin{array}{|c|}\hline \sigma\rho = (2,4)(1,2,3,4) = (1,4)(2,3), \text{ which is reflection across } y = -x \\ \sigma\rho^2 = (2,4)(1,3)(2,4) = (1,3), \text{ which is reflection across the } y\text{-axis} \\ \sigma\rho^3 = (2,4)(1,4,3,2) = (1,2)(3,4), \text{ which is reflection across } y = x \\ \hline \end{array}$$

   (b) **Complete the following multiplication table for D_4. All entries should be of the form (0.0.1).**

	1	ρ	ρ^2	ρ^3	σ	$\sigma\rho$	$\sigma\rho^2$	$\sigma\rho^3$
1								
ρ								
ρ^2								
ρ^3								
σ								
$\sigma\rho$								
$\sigma\rho^2$								
$\sigma\rho^3$								

One may fill in the chart using only the facts that

$$\sigma^2 = 1, \quad \rho^4 = 1, \quad \rho\sigma = \sigma\rho^3 :$$

	1	ρ	ρ^2	ρ^3	σ	$\sigma\rho$	$\sigma\rho^2$	$\sigma\rho^3$
1	1	ρ	ρ^2	ρ^3	σ	$\sigma\rho$	$\sigma\rho^2$	$\sigma\rho^3$
ρ	ρ	ρ^2	ρ^3	1	$\sigma\rho^3$	σ	$\sigma\rho$	$\sigma\rho^2$
ρ^2	ρ^2	ρ^3	1	ρ	$\sigma\rho^2$	$\sigma\rho^3$	σ	$\sigma\rho$
ρ^3	ρ^3	1	ρ	ρ^2	$\sigma\rho$	$\sigma\rho^2$	$\sigma\rho^3$	σ
σ	σ	$\sigma\rho$	$\sigma\rho^2$	$\sigma\rho^3$	1	ρ	ρ^2	ρ^3
$\sigma\rho$	$\sigma\rho$	$\sigma\rho^2$	$\sigma\rho^3$	σ	ρ^3	1	ρ	ρ^2
$\sigma\rho^2$	$\sigma\rho^2$	$\sigma\rho^3$	σ	$\sigma\rho$	ρ^2	ρ^3	1	ρ
$\sigma\rho^3$	$\sigma\rho^3$	σ	$\sigma\rho$	$\sigma\rho^2$	ρ	ρ^2	ρ^3	1