

Math 701, Spring 2007, Final Exam
The exam ends at 1:00 PM.

If I am not here when you finish, then please bring your solutions to my office – 300B.

Note! You must show sufficient work to support your answer. Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc. Leave room in the upper left hand corner for the staple. If you are asked to prove something that is false, then provide a counterexample instead.

Each problem is worth 10 points.

1. **Let G be a group of order p^n for some prime p and let H be a normal subgroup of G , with $H \neq \{1\}$. Prove that $Z(G) \cap H \neq \{1\}$, where $Z(G)$ is the center of G .**

Let G act on H by conjugation. It follows that

$$|H| = |\{h \in H \mid ghg^{-1} = h \text{ for all } g \in G\}| + \sum_x [G : \text{stab}(x)],$$

where the sum is taken over all orbits of size more than 1 and we take one element x from each orbit. Observe that

$$\{h \in H \mid ghg^{-1} = h \text{ for all } g \in G\} = Z(G) \cap H.$$

We see that p divides the order of H and p divides each summand $[G : \text{stab}(x)]$. So p divides $|Z(G) \cap H|$. Of course, the identity element is in $Z(G) \cap H$; so this subgroup must have order at least p .

2. **TRUE or FALSE. (If true, PROVE it. If false, give a COUNTEREXAMPLE.) If H and K are normal subgroups of the finite group G , with $H \cong K$, then $\frac{G}{H} \cong \frac{G}{K}$.**

FALSE. Let $G = \frac{\mathbb{Z}}{(4)} \oplus \frac{\mathbb{Z}}{(2)}$, $H = \langle (2, 0) \rangle$, and $K = \langle (0, 1) \rangle$. Every subgroup of G is normal. The subgroups H and K are isomorphic because they each are cyclic of order two. The quotients $\frac{G}{H} \cong \frac{\mathbb{Z}}{(2)} \oplus \frac{\mathbb{Z}}{(2)}$ and $\frac{G}{K} \cong \frac{\mathbb{Z}}{(4)}$ are not isomorphic.

3. Recall that the *exponent* of the abelian group $(A, +)$ is the least positive integer n such that $na = 0$ for all a in A .
- (a) Let p be a prime integer and let G_1 and G_2 be abelian groups with p^6 elements. Suppose that G_1 and G_2 have the same exponent and the same number of elements of order p . Prove that G_1 and G_2 are isomorphic. Show that the conclusion would not hold if G_1 and G_2 both have p^7 elements.

There is a one-to-one correspondence between the set of partitions of 6 and the set isomorphism classes of abelian groups with p^6 elements. The correspondence is given by

$$(**) \quad n_1 \geq n_2 \cdots \geq n_\ell \leftrightarrow \frac{\mathbb{Z}}{(p^{n_1})} \oplus \cdots \oplus \frac{\mathbb{Z}}{(p^{n_\ell})}$$

The group in $(**)$ has exponent p^{n_1} and $p^\ell - 1$ elements of order p . There are 11 partitions of 6. A quick glance shows that it is never the case that two partitions of 6 have the same largest part and also the same number of parts:

$$\begin{array}{l}
 6 \\
 5, 1 \\
 4, 2 \\
 4, 1, 1 \\
 3, 3 \\
 (***) \quad 3, 2, 1 \\
 3, 1, 1, 1 \\
 2, 2, 2 \\
 2, 2, 1, 1 \\
 2, 1, 1, 1, 1 \\
 1, 1, 1, 1, 1, 1.
 \end{array}$$

The groups

$$\frac{\mathbb{Z}}{(p^3)} \oplus \frac{\mathbb{Z}}{(p^3)} \oplus \frac{\mathbb{Z}}{(p)} \quad \text{and} \quad \frac{\mathbb{Z}}{(p^3)} \oplus \frac{\mathbb{Z}}{(p^2)} \oplus \frac{\mathbb{Z}}{(p^2)}$$

are non-isomorphic groups of order p^7 . Both groups have exponent p^3 and $p^3 - 1$ elements of order p .

- (b) Let N_1 and N_2 be 6×6 nilpotent matrices over \mathbb{C} . Suppose that N_1 and N_2 have the same minimal polynomial and the same nullity. Prove that N_1 and N_2 are similar. Show that this is not true for 7×7 nilpotent matrices.

For each positive integer t , let J_t be the $t \times t$ matrix with 1 in position $(i+1, i)$ for $1 \leq i \leq t-1$, and zero everywhere else:

$$J_1 = [0], \quad J_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{etc.}$$

Notice that the nullity of each J_t is 1.

There is a one-to-one correspondence between the set of partitions of 6 and the set of similarity classes of nilpotent 6×6 matrices. The correspondence is given by

$$(*) \quad n_1 \geq n_2 \cdots \geq n_\ell \leftrightarrow \begin{bmatrix} J_{n_1} & 0 & \cdots & 0 \\ 0 & J_{n_2} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & J_{n_\ell} \end{bmatrix}$$

The matrix in (*) has minimal polynomial x^{n_1} and nullity ℓ . There are 11 partitions of 6. A quick glance at (***) continues to show that it is never the case that two partitions of 6 have the same largest part and also the same number of parts.

On the other hand the following 7×7 matrices are not similar, but both have minimal polynomial x^3 and nullity 3:

$$\begin{bmatrix} J_3 & 0 & 0 \\ 0 & J_3 & 0 \\ 0 & 0 & J_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} J_3 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_1 \end{bmatrix}.$$

4.

- (a) **Give an example of a non-zero prime ideal which is not a maximal ideal. Prove your answer.**

The ideal (x) in $\mathbb{Q}[x, y]$ is a prime ideal (because $\frac{\mathbb{Q}[x, y]}{(x)}$ is the domain $\mathbb{Q}[y]$) but is not a maximal ideal (because (x) is a proper subideal of (x, y) .)

- (b) **Let I be a non-zero prime ideal in the Principal Ideal Domain R . Prove that I is a maximal ideal of R .**

The ideal I is principal, generated by a . Let J be an ideal of R with $I \subsetneq J$. The ring R is a PID; so, $J = (b)$ for some $b \in R$, and $a = rb$ for some $r \in R$. The hypothesis that $I \neq J$ ensures that $b \notin I$. The hypothesis that I is a prime ideal ensures that $r \in I$; hence, $r = sa$ for some $s \in R$. We now have $a = sab$. The ring R is a domain; hence, $1 = sb$, b is a unit of R , and J is necessarily equal to all of R .

5. Let M be a module over the Principal Ideal Domain R . Suppose that M is generated, as an R -module, by m_1 and m_2 and that the annihilator of m_i is I_i . (Recall that the annihilator of the element m in the R -module M is the ideal $\{r \in R \mid rm = 0\}$.) Suppose further that $(m_1) \cap (m_2) = 0$, where (m) is the R -submodule of M generated by m . Let p be an irreducible element of R .

(a) Identify a generating set for the R -module $\frac{M}{pM}$. What is the annihilator of each generator? Prove your answer.

If m is in M , then let \overline{m} be the image of m in $\frac{M}{pM}$. It is clear that $\frac{M}{pM}$ is generated by $\overline{m_1}$ and $\overline{m_2}$.

Claim: There are two possibilities for the annihilator of $\overline{m_i}$ in $\frac{M}{pM}$. Either $\overline{m_i}$ is zero in $\frac{M}{pM}$ (in which case, the annihilator of $\overline{m_i}$ is R) or $\overline{m_i}$ is not zero in $\frac{M}{pM}$ (in which case, the annihilator of $\overline{m_i}$ is (p)).

Proof. Fix i . The maximal ideal (p) of R is contained in $\text{ann} \overline{m_i}$. So either $\text{ann} \overline{m_i} = (p)$ or $\text{ann} \overline{m_i} = R$ (and in this case $\overline{m_i} = 0$).

(b) Identify a generating set for the R -module $\frac{pM}{p^2M}$. What is the annihilator of each generator? Prove your answer.

If m is in M , NOW let \overline{m} be the image of m in $\frac{pM}{p^2M}$. It is clear that $\frac{pM}{p^2M}$ is generated by $\overline{pm_1}$ and $\overline{pm_2}$.

Claim: There are two possibilities for the annihilator of $\overline{pm_i}$ in $\frac{pM}{p^2M}$. Either $\overline{pm_i}$ is zero in $\frac{pM}{p^2M}$ (in which case, the annihilator of $\overline{pm_i}$ is R) or $\overline{pm_i}$ is not zero in $\frac{pM}{p^2M}$ (in which case, the annihilator of $\overline{pm_i}$ is (p)).

Proof. Fix i . The maximal ideal (p) of R is contained in $\text{ann} \overline{pm_i}$. So either $\text{ann} \overline{pm_i} = (p)$ or $\text{ann} \overline{pm_i} = R$ (and in this case $\overline{pm_i} = 0$).

6. In a commutative ring R , let I be an ideal which is maximal in the set of all non-principal ideals of R . Prove that I is a prime ideal.

Proof by contradiction. Suppose that I is not a prime ideal of R . Then there exist elements a, b of R with $a \notin I$, $b \notin I$, and $ab \in I$. The ideal (I, a) properly contains I ; so (I, a) is a principal ideal, that is $(I, a) = (c)$ for some $c \in R$. Consider the ideal $I:(c)$ which is defined to be $\{r \in R \mid rc \in I\}$. We know that $(I, b) \subseteq I:(c)$; hence, $I:(c)$ is an ideal which properly contains I . In other words, there is an element d of R with $(d) = I:(c)$. We reach a contradiction by showing that I is the principal ideal (cd) . It is clear that $(cd) \subseteq I$. On the other hand, if $i \in I$, then i is in $(I, a) = (c)$; so $i = rc$ for some $r \in R$. This element r is obviously in $I:(c) = (d)$. In other words, $i = sdc$ for some s in R .