FINAL EXAM MATH 701 FALL 2023

Write your answers as **legibly** as you can on the blank sheets of paper provided. Write **complete** answers in **complete sentences**. Make sure that your **notation is defined**!

Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

If some problem is incorrect, then give a counterexample and/or supply the missing hypothesis and prove the resulting statement. If some problem is vague, then be sure to explain your interpretation of the problem.

You should KEEP this piece of paper. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 100 points. There are nine problems.

1. (11 points) Prove that there is no finite group G with $G/Z(G)$ has exactly 143 elements.
2. (11 points) Let U be the multiplicative group of complex numbers of modulus 1, let $\mathbb{R}$ be the additive group of real numbers, and let $\mathbb{Z}$ be the additive group of integers. Prove that $U \cong \mathbb{R}/\mathbb{Z}$.
3. (11 points) Let G_1 and G_2 be Abelian groups and let $\alpha : G_1 \to G_2$ and $\beta : G_2 \to G_1$ be group homomorphisms so that $\beta\alpha(g) = g$, for all $g \in G_1$. Prove that G_2 is isomorphic to $\operatorname{im} \alpha \oplus \ker \beta$.
4. (11 points) Consider $\alpha : \frac{\mathbb{Z}}{\langle 9 \rangle} \to \frac{\mathbb{Z}}{\langle 18 \rangle}$, given by $\bar{n} \mapsto \bar{n}$, and $\beta : \frac{\mathbb{Z}}{\langle 18 \rangle} \to \frac{\mathbb{Z}}{\langle 9 \rangle}$ given by $\bar{n} \mapsto \bar{n}$. Is α a group homomorphism? Is β a group homomorphism? Explain.
5. (11 points) Let R be a commutative ring. Let I be a prime ideal of R such that R/I satisfies the descending chain condition on ideals. Prove that R/I is a field.
6. (11 points) Let $R = \mathbb{Z}[x]$. Give three prime ideals of R that contain the ideal $(6, 2x)$.
7. (11 points) Prove the following form of the Chinese Remainder Theorem. Let R be a commutative ring and suppose that I and J are ideals of R such that $I + J = R$. Then $\frac{R}{I \cap J}$ and $\frac{R}{I} \oplus \frac{R}{J}$ are isomorphic rings. (The direct sum of two rings is a ring; the multiplication takes place coordinate-wise.)
8. (12 points) Let R be a commutative ring. For $x \in R$, let $A(x) = \{r \in R \mid xr = 0\}$. Suppose $\theta \in R$ has the property that $A(\theta)$ is not properly contained in $A(x)$ for any $x \in R$. Prove the $A(\theta)$ is a prime ideal of R.
9. (11 points) Let p be the smallest prime dividing the order of the finite group G. Prove that any subgroup of index p in G is a normal subgroup.