# EXAM 2 MATH 701 FALL 2023

Write your answers as **legibly** as you can on the blank sheets of paper provided. Write **complete** answers in **complete sentences**. Make sure that your **notation is defined**!

Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

If some problem is incorrect, then give a counterexample and/or supply the missing hypothesis and prove the resulting statement. If some problem is vague, then be sure to explain your interpretation of the problem.

**You should KEEP this piece of paper.** Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points.

1. Let $G$ and $H$ be groups. Each of the following three parts is a True/False question. If the statement is true, prove it. If the statement is false, give a counterexample.
   (a) (5 points) True or False. If $\phi : G \to H$ is a surjective group homomorphism then $G$ is isomorphic to $H \oplus \ker \phi$.
   (b) (5points) True or False. If $\phi : G \to H$ and $i : H \to G$ are group homomorphisms with $\phi \circ i$ equal to the identity map on $H$, then $G$ is isomorphic to $H \oplus \ker \phi$.
   (c) (5 points) True or False. If $G$ is an Abelian group and $\phi : G \to H$ and $i : H \to G$ are group homomorphisms with $\phi \circ i$ equal to the identity map on $H$, then $G$ is isomorphic to
   $H \oplus \ker \phi$.
2. (11 points) Prove that there are no simple groups of order 72.
3. (a) (2 points) Let $\phi : G \to G'$ be a group homomorphism. What quotient of $G$ is isomorphic to the image of $\phi$?
   (b) (8 points) Let $H$ and $N$ be subgroups of a group $G$ with $N$ a normal subgroup of $G$. State and prove the Second Isomorphism Theorem. (This is the result which establishes an isomorphism between $\frac{HN}{N}$ and some quotient of $H$.) You may appeal to the result stated in (a).
4. Recall that the *annihilator* of an additive Abelian group $G$ is the least positive integer $N$ with $Ng = 0$ for all $g \in G$.
   (a) (7 points) Let $G$ and $G'$ be Abelian groups of order $p^6$ for some prime integer $p$. Suppose that both sets
   $$\{g \in G \mid pg = 0\} \quad \text{and} \quad \{g' \in G' \mid pg' = 0\}$$
   have $p^3$ elements and each group has annihilator $p^3$. Must $G$ and $G'$ be isomorphic? Give a proof if the groups must be isomorphic and an example if the groups need not be isomorphic.
   (b) (7 points) Let $G$ and $G'$ be Abelian groups of order $p^7$ for some prime integer $p$. Suppose that both sets
   $$\{g \in G \mid pg = 0\} \quad \text{and} \quad \{g' \in G' \mid pg' = 0\}$$
   have $p^3$ elements and each group has annihilator $p^3$. Must $G$ and $G'$ be isomorphic? Give a proof if the groups must be isomorphic and an example if the groups need not be isomorphic.