EXAM 2 MATH 701 FALL 2023

Write your answers as **legibly** as you can on the blank sheets of paper provided. Write **complete** answers in **complete sentences**. Make sure that your **notation is defined**!

Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

If some problem is incorrect, then give a counterexample and/or supply the missing hypothesis and prove the resulting statement. If some problem is vague, then be sure to explain your interpretation of the problem.

You should KEEP this piece of paper. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points.

1. **Let G and H be groups. Each of the following three parts is a True/False question. If the statement is true, prove it. If the statement is false, give a counterexample.**
   (a) (5 points) **True or False. If $\phi : G \to H$ is a surjective group homomorphism then G is isomorphic to $H \oplus \ker \phi$.**

   False. The homomorphism $S_3 \to \frac{S_3}{A_3}$ is surjective; but S_3 is not isomorphic to $A_3 \oplus \frac{S_3}{A_3}$ because $A_3 \oplus \frac{S_3}{A_3}$ is a cyclic group of order 6 and S_3 is not cylic.

   (b) (5points) **True or False. If $\phi : G \to H$ and $i : H \to G$ are group homomorphisms with $\phi \circ i$ equal to the identity map on H, then G is isomorphic to $H \oplus \ker \phi$.**

   False. Define $\pi : S_4 \to \frac{S_4}{V_4}$ to be the natural quotient map and $i : S_3 \to S_4$ be the natural inclusion map. Observe that $\pi \circ i : S_3 \to \frac{S_4}{V_4}$ is an isomorphism. Let $\phi : S_4 \to S_3$ be the composition $(\pi \circ i)^{-1} \circ \pi$. It is clear that $\phi \circ i$ is the identity map on S_3. But S_4 is not isomorphic to the direct sum of S_3 and any group of ordedr 4. In particular, S_4 does not have a normal subgroup of order six.

   (c) (5 points) **True or False. If G is an Abelian group and $\phi : G \to H$ and $i : H \to G$ are group homomorphisms with $\phi \circ i$ equal to the identity map on H, then G is isomorphic to**
   $H \oplus \ker \phi$.

   True. Indeed $G = \ker \phi \oplus \operatorname{im} i$. We show that $\ker \phi \cap \operatorname{im} i = 0$ and $\ker \phi + \operatorname{im} i = G$.

   $\ker \phi \cap \operatorname{im} i = 0$: If $x \in \ker \phi \cap \operatorname{im} i$, then $x = i(h)$ for some $h \in H$ and $h = \phi(i(h)) = \phi(x) = 0$; hence $x = 0$.

$\ker\phi + \operatorname{im} i = G$: If $x \in G$, then $x = (i\circ\phi)(x) + (x - (i\circ\phi)x)$, with $(i\circ\phi)(x) \in \operatorname{im} i$ and $(x - (i\circ\phi)x) \in \ker\phi$.

2. (11 points) **Prove that there are no simple groups of order 72.**

Suppose G is a simple group of order 72. Let n_3 be the number of Sylow subgroups of G or order 3. We know that $n_3 \neq 1$, n_3 is congruent to 1 mod 3, and n_3 divides 8. Thus, $n_3 = 4$. The group G acts on the set of Sylow three subgroups of G by conjugation. Thus, there is a group homomorphism ϕ from G to S_4. The group G has more elements than S_4. So, there are elements in $\ker\phi$ in addition to the identity element. Thus, $\ker\phi$ is a non-trivial normal subgroup of G.

3. (a) (2 points) **Let $\phi : G \to G'$ be a group homomorphism. What quotient of G is isomorphic to the image of ϕ?**

The first isomorphism theorem guarantees that $\frac{G}{\ker\phi}$ is isomorphic to $\operatorname{im}\phi$.

(b) (8 points) **Let H and N be subgroups of a group G with N a normal subgroup of G. State and prove the Second Isomorphism Theorem. (This is the result which establishes an isomorphism between $\frac{HN}{N}$ and some quotient of H.) You may appeal to the result stated in (a).**

$\frac{HN}{N} \cong \frac{H}{H\cap N}$.

Proof. Consider the composition $H \to HN \to \frac{HN}{N}$, where the first map is the natural innclusion map and the second map is the natural quotient map. It is clear that the composition is surjective. It is also clear the the kernel is $H \cap N$. The conclusion follows from part (a). □

4. **Recall that the *annihilator* of an additive Abelian group G is the least positive integer N with $Ng = 0$ for all $g \in G$.**

(a) (7 points) **Let G and G' be Abelian groups of order p^6 for some prime integer p. Suppose that both sets**

$$\{g \in G \mid pg = 0\} \quad \text{and} \quad \{g' \in G' \mid pg' = 0\}$$

have p^3 elements and each group has annihilator p^3. Must G and G' be isomorphic? Give a proof if the groups must be isomorphic and an example if the groups need not be isomorphic.

According to the structure theorem for finitely generated Abelian groups there are positive integers $a_1 \le \cdots \le a_\ell$ and $b_1 \le \cdots \le b_m$ such that

$$G = \frac{\mathbb{Z}}{p^{a_1}} \oplus \frac{\mathbb{Z}}{p^{a_2}} \oplus \ldots \oplus \frac{\mathbb{Z}}{p^{a_\ell}}$$

and

$$G' = \frac{\mathbb{Z}}{p^{b_1}} \oplus \frac{\mathbb{Z}}{p^{b_2}} \oplus \ldots \oplus \frac{\mathbb{Z}}{p^{b_m}}.$$

The hypothesis about $pg = 0$ ensures that $\ell = m = 3$. The hypothesis about annihilator ensures that $a_3 = b_3 = 3$. The hypothesis about the order of the groups ensures that $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 6$. It is now clear that (a_1, a_2) and (b_1, b_2) must both be $(1, 2)$.

(b) (7 points) **Let G and G' be Abelian groups of order p^7 for some prime integer p. Suppose that both sets**

$$\{g \in G \mid pg = 0\} \quad \textbf{and} \quad \{g' \in G' \mid pg' = 0\}$$

have p^3 elements and each group has annihilator p^3. Must G and G' be isomorphic? Give a proof if the groups must be isomorphic and an example if the groups need not be isomorphic.

We need integers $1 \le a_1 \le a_2 \le a_3$ and $1 \le b_1 \le b_2 \le b_3$ with $a_3 = b_3 = 3$ and $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 7$. This problem has two solutions; namely $(1, 3, 3)$ and $(2, 2, 3)$. That is, the non–isomorphic groups

$$\frac{\mathbb{Z}}{p^1\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^3\mathbb{Z}} \quad \text{and} \quad \frac{\mathbb{Z}}{p^2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^3\mathbb{Z}}$$

both satisfy the hypotheses.