EXAM 1 MATH 701 FALL 2023

Write your answers as **legibly** as you can on the blank sheets of paper provided. Write **complete** answers in **complete sentences**. Make sure that your **notation is defined**!

Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

If some problem is incorrect, then give a counterexample and/or supply the missing hypothesis and prove the resulting statement. If some problem is vague, then be sure to explain your interpretation of the problem.

The symbols $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ represent the ring of integers, the field of real numbers, and the field of complex numbers, respectively. The elements of $\mathbb{R}^n$ are column vectors with n-entries. A subgroup V of $\mathbb{R}^n$ is a subspace if V is closed under scalar multiplication by elements of $\mathbb{R}$.

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points.

1. (9 points) Let M be a real symmetric $n \times n$ matrix. Suppose that v is a non-zero vector and λ is a real number with $Mv = \lambda v$. Prove that there is a subspace W of $\mathbb{R}^n$ so that $\mathbb{R}^n = \mathbb{R}v \oplus W$ with $MW \subseteq W$. (Please prove the statement directly. Do not deduce the statement from a more sophisticated result.)
2. (8 points) Consider $\phi : \frac{\mathbb{Z}}{\langle 8 \rangle} \to \frac{\mathbb{Z}}{\langle 12 \rangle}$, given by $\phi(n+\langle 8\rangle) = n+\langle 12\rangle$. Is ϕ a group homomorphism? Explain thoroughly.
3. (9 points) State and prove the Chinese Remainder Theorem.
4. (8 points) Let G be a group which has exactly one element g of order n , where n is a positive integer. Prove that $n = 2$ and g is in the center of G. (Recall that the center of G is the set of all elements in G that commute with all elements of G.)
5. (8 points) Let U be the unit circle subgroup of $(\mathbb{C} \setminus \{0\}, \times)$ and, for each positive integer n, let U_n the subgroup of n^{th} roots in U. Prove that the groups $\frac{U}{U_8}$ and $\frac{U}{U_{28}}$ are isomorphic.
6. (8 points) List as many non-isomorphic groups of order eight as you can. Explain why none of the groups on your list are isomorphic to any other group on your list.