EXAM 1 MATH 701 FALL 2023

Write your answers as **legibly** as you can on the blank sheets of paper provided. Write **complete** answers in **complete sentences**. Make sure that your **notation is defined**!

Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

If some problem is incorrect, then give a counterexample and/or supply the missing hypothesis and prove the resulting statement. If some problem is vague, then be sure to explain your interpretation of the problem.

The symbols $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ represent the ring of integers, the field of real numbers, and the field of complex numbers, respectively. The elements of $\mathbb{R}^n$ are column vectors with n-entries. A subgroup V of $\mathbb{R}^n$ is a <u>subspace</u> if V is closed under scalar multiplication by elements of $\mathbb{R}$.

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points.

1. (9 points) **Let M be a real symmetric $n \times n$ matrix. Suppose that v is a non-zero vector and λ is a real number with $Mv = \lambda v$. Prove that there is a subspace W of $\mathbb{R}^n$ so that $\mathbb{R}^n = \mathbb{R}v \oplus W$ with $MW \subseteq W$. (Please prove the statement directly. Do not deduce the statement from a more sophisticated result.)**

   Let $W = \{w \in \mathbb{R}^n \mid w^{\mathrm{T}} v = 0\}$.

   We show that $MW \subseteq W$. If $w \in W$, then
   $$(Mw)^{\mathrm{T}} v = w^{\mathrm{T}} M^{\mathrm{T}} v = w^{\mathrm{T}} M v = w^{\mathrm{T}} \lambda v = \lambda w^{\mathrm{T}} v = 0.$$
   It is clear that $\mathbb{R}v \cap W = (0)$.
   We show that $\mathbb{R}v + W = \mathbb{R}^n$.
   The direction $\subseteq$ is obvious. We show $\supseteq$. Take $u \in \mathbb{R}^n$. Observe that
   $$u = \frac{v^{\mathrm{T}} u}{v^{\mathrm{T}} v} v + (u - \frac{v^{\mathrm{T}} u}{v^{\mathrm{T}} v} v).$$
   It is clear that $\frac{v^{\mathrm{T}} u}{v^{\mathrm{T}} v} v \in \mathbb{R}v$. We show $(u - \frac{v^{\mathrm{T}} u}{v^{\mathrm{T}} v} v) \in W$. Observe that
   $$(u - \frac{v^{\mathrm{T}} u}{v^{\mathrm{T}} v} v)^{\mathrm{T}} v = u^{\mathrm{T}} v - \frac{v^{\mathrm{T}} u}{v^{\mathrm{T}} v} v^{\mathrm{T}} v = 0.$$

2. (8 points) **Consider $\phi : \frac{\mathbb{Z}}{\langle 8 \rangle} \to \frac{\mathbb{Z}}{\langle 12 \rangle}$, given by $\phi(n + \langle 8 \rangle) = n + \langle 12 \rangle$. Is ϕ a group homomorphism? Explain thoroughly.**

   NO! This "ϕ" is not even a function! $1 + \langle 8 \rangle = 9 + \langle 8 \rangle 12$; but
   $$\text{“}\phi\text{”}(1 + \langle 8 \rangle) = (1 + \langle 12 \rangle) \neq (9 + \langle 12 \rangle) = \text{“}\phi\text{”}(9 + \langle 12 \rangle).$$

3. (9 points) **State and prove the Chinese Remainder Theorem.**

   THEOREM. If r and s are relatively prime integers, then $\frac{\mathbb{Z}}{\langle r\rangle}\oplus\frac{\mathbb{Z}}{\langle s\rangle}$ is isomorphic to $\frac{\mathbb{Z}}{\langle rs\rangle}$.

   Define $\phi:\mathbb{Z}\to\frac{\mathbb{Z}}{\langle r\rangle}\oplus\frac{\mathbb{Z}}{\langle s\rangle}$ by $\phi(n)=(n+\langle r\rangle, n+\langle s\rangle)$. The integers r and s are relatively prime; hence, there are integers a and b so that $ar+bs=1$. Observe that $\phi(1-ar)=(1+\langle r\rangle,0)$ and $\phi(1-bs)=(0,1+<s>)$. It follows that ϕ is surjective. The source and the target both have rs elements; hence ϕ is a bijection.

4. (8 points) **Let G be a group which has exactly one element g of order n , where n is a positive integer. Prove that $n=2$ and g is in the center of G. (Recall that the center of G is the set of all elements in G that commute with all elements of G.)**

   It is always true that g and the inverse of g have have the same order. If g is the only element of order n, then g must equal g^{-1}; hence n must equal 2. Observe that g and $h^{-1}gh$ always have the same order. If g is the only element of order 2, then g and $h^{-1}gh$ must be equal; hence, $gh=hg$ for all $h\in G$.

5. (8 points) **Let U be the unit circle subgroup of $(\mathbb{C}\setminus\{0\},\times)$ and, for each positive integer n, let U_n the subgroup of n^{th} roots in U. Prove that the groups $\frac{U}{U_8}$ and $\frac{U}{U_{28}}$ are isomorphic.**

   We prove that both group are isomorphic to U. Apply the first isomorphism theorem to the map $U\to U$, which sends u to u^8, for $u\in U$, to conclude that $\frac{U}{U_8}$ is isomorphic to U. In a similar manner, apply the first isomorphism theorem to the map $U\to U$, which sends u to u^{28}, for $u\in U$, to conclude that $\frac{U}{U_{28}}$ is isomorphic to U.

6. (8 points) **List as many non-isomorphic groups of order eight as you can. Explain why none of the groups on your list are isomorphic to any other group on your list.**

   The groups of order 8 are $\frac{\mathbb{Z}}{\langle 8\rangle}$, $\frac{\mathbb{Z}}{\langle 4\rangle}\oplus\frac{\mathbb{Z}}{\langle 2\rangle}$, $\frac{\mathbb{Z}}{\langle 2\rangle}\oplus\frac{\mathbb{Z}}{\langle 2\rangle}\oplus\frac{\mathbb{Z}}{\langle 2\rangle}$, D_4, Q_8. The first three groups are Abelian. Groups four and five are not Abelian. The fifth group has exactly one element of order 2. The fourth group has many elements or order 2. The first group has an element of order eight; groups two and three do not. The second group has an element of order 4; the third group does not.