Final Exam 1991

Each problem is worth 25 points.

1. Give an example of two nilpotent 4×4 matrices that have the same minimal polynomial, but are not similar. (Recall that the matrix A is *nilpotent* if $A^r = 0$ for some integer r.)

2. Suppose $T\colon \mathbb{R}^5 \to \mathbb{R}^5$ is a linear transformation which is represented in some basis by a diagonal matrix with entries $2, 2, 4, 4, 4$ on the main diagonal. What is the rational canonical form for T?

3. Suppose that $T\colon V \to V$ is a linear transformation on a finite dimensional vector space. If T has rank one, then prove that T is nilpotent or diagonalizable.

4. Let M be an $n \times n$ matrix over the field of Complex numbers. If r denotes the rank of M, then prove that M can be written as the sum of r rank one matrices.

5. Give an example of a vector space V, a linear transformation $T\colon V \to V$, and a $T-$invariant subspace W of V such that no complement of W in V is $T-$invariant. Prove that your example does what it is supposed to do. (Note: The subspace W' of V is a complement of W in V if $W \oplus W' = V$.)

6. Let V and W be finite dimensional vector spaces over the field F and let $T\colon V \to W$ be a linear transformation. Fix a basis $\mathcal{B}\colon v_1, \ldots, v_n$ for V and a basis $\mathcal{C}\colon w_1, \ldots, w_m$ for W. For each vector $v \in V$ and $w \in W$ define

$$[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \text{and} \quad [w]_{\mathcal{C}} = \begin{bmatrix} c_1' \\ \vdots \\ c_m' \end{bmatrix}$$

   where $v = \sum_{i=1}^{n} c_i v_i$ and $w = \sum_{i=1}^{m} c_i' w_i$ for scalars c_i and c_i' in F.
   (a) Define the matrix $[T]_{\mathcal{C}\,\mathcal{B}}$ which represents the transformation T with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.
   (b) Let $T^*\colon W^* = \mathrm{Hom}_F(W, F) \to V^* = \mathrm{Hom}_F(V, F)$ be the dual of T. Let $\mathcal{B}^*$ be the basis for V^* which is dual to $\mathcal{B}$ and $\mathcal{C}^*$ be the basis for W^* which is dual to $\mathcal{C}$. How are the matrices $[T]_{\mathcal{C}\,\mathcal{B}}$ and $[T^*]_{\mathcal{B}^*\,\mathcal{C}^*}$ related?
   (c) Prove your answer to part (b). Be sure to justify each step.

7. Suppose that $T\colon V \to V$ is a linear transformation on a finite dimensional vector space V over the field F. Prove that V is $T-$cyclic if and only if

$$\{U \in \mathrm{Hom}_F(V, V) : TU = UT\} = \{f(T) : f \in F[x]\}.$$