MATH 700
HOMEWORK 6

Due Friday, October 11, 1991 at the beginning of class.

Definition. If V is a vector space over the field F, then the *dual of* V is the vector space $V^* = \mathrm{Hom}_F(V, F)$. If V and W are vector spaces over the field F, and $T\colon V \to W$ is a linear transformation, then the *dual of* T is the transformation $T^*\colon W^* \to V^*$ which is defined by $T^*(\varphi) = \varphi \circ T$ for all $\varphi \in W^*$.

Definition. If $A = (a_{ij})$ is an $n \times n$ matrix with entries in the field F, then the *trace* of A is $\sum_{i=1}^{n} a_{ii}$.

1. (Hoffman and Kunze page 116, number 6) Let V be the vector space of polynomials of degree at most n over $\mathbb{R}$. Let $D\colon V \to V$ be differentiation. Find a basis for the null space of D^*.

2. (Hoffman and Kunze page 116, number 7) Let V be a finite dimensional vector space over the field F. Let $\varphi\colon \mathrm{Hom}_F(V, V) \to \mathrm{Hom}_F(V^*, V^*)$ be the function which is defined by $\varphi(T) = T^*$ for all $T \in \mathrm{Hom}_F(V, V)$. Prove that φ is an isomorphism of vector spaces.

3. (Hoffman and Kunze page 116, number 8) Let V be the vector space of $n \times n$ matrices over the field F.
   (a) If $B \in V$, then define the function $f_B\colon V \to F$ by $f_B(A) = \mathrm{trace}(B^{\mathrm{t}} A)$. Prove that f_B is a linear transformation. (In this problem Bt is the transpose of the matrix B.)
   (b) Let $\Phi\colon V \to V^*$ be the function defined by $\Phi(B) = f_B$ for all $B \in V$. Prove that Φ is onto.
   (c) Prove that the function Φ of (b) is an isomorphism.