Exam 1, 1991

Each problem is worth 25 points.

1. Fill in the blank and prove the resulting statement. If V_1 and V_2 are subspaces of a vector space V, then $V_1 \cup V_2$ is a subspace of V if and only if _____.

2. Let $T\colon V_1 \to V_2$ and $S\colon V_3 \to V_2$ be linear transformations of vector spaces. If T is onto, then prove that there exists a linear transformation $\varphi\colon V_3 \to V_1$ such that $T \circ \varphi = S$.

3. Let V and V' be finite dimensional vector spaces over the field F and let $T\colon V \to V'$ be a linear transformation. Fix a basis $\mathcal{B}\colon v_1, \ldots, v_n$ for V and a basis $\mathcal{B}'\colon v'_1, \ldots, v'_m$ for V'. For each vector $v \in V$ and $v' \in V'$ define

$$[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \text{and} \quad [v']_{\mathcal{B}'} = \begin{bmatrix} c'_1 \\ \vdots \\ c'_m \end{bmatrix}$$

where $v = \sum_{i=1}^{n} c_i v_i$ and $v' = \sum_{i=1}^{m} c'_i v'_i$ for scalars c_i and c'_i in F.

(a) Define the matrix $[T]_{\mathcal{B}'\,\mathcal{B}}$ which represents the transformation T with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$.

(b) What formula relates $[v]_{\mathcal{B}}$, $[T(v)]_{\mathcal{B}'}$, and $[T]_{\mathcal{B}'\,\mathcal{B}}$ for all $v \in V$?

(c) Prove your answer to part (b). Be sure to justify each step.

4. Let V be a finite dimensional vector space over the field F, let W be a subspace of V and let $\operatorname{ann}(W)$ be the following subspace of V^*:

$$\operatorname{ann}(W) = \{\varphi \in \operatorname{Hom}_F(V, F) \mid \varphi(w) = 0 \text{ for all } w \in W\}.$$

(a) What formula relates $\dim W$, $\dim \operatorname{ann}(W)$ and $\dim V$?

(b) Prove your answer to part (a).