Final Exam, 1989

There are 9 problems. Each problem is worth 20 points.

1. Fill in the blank and then prove the resulting statement. Let W_1 and W_2 be subspaces of a vector space V. The subset $W_1 \cup W_2$ is a subspace of V if and only if ________________ .

2. Suppose that
$$0 \xrightarrow{f_5} V_4 \xrightarrow{f_4} V_3 \xrightarrow{f_3} V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} 0$$
is an exact sequence of linear transformations. (In other words, for each i, $f_i \colon V_i \to V_{i-1}$ is a linear transformation and $\ker f_i = \operatorname{im} f_{i+1}$.) Suppose further, that each V_i is finite dimensional.
(a) Give a formula which relates the dimensions of the V_i.
(b) Prove your answer to (a).

3. If $0 \to V \xrightarrow{S} W \xrightarrow{T} Z \to 0$ is an exact sequence of vector spaces over the field F, then prove that
$$0 \to \operatorname{Hom}_F(Z, F) \xrightarrow{T^*} \operatorname{Hom}_F(W, F) \xrightarrow{S^*} \operatorname{Hom}_F(V, F) \to 0$$
is also an exact sequence of vector space. (Note: The notion of exact sequence is defined in problem 2.)

4. Let M be a square matrix with entries in the arbitrary field F. Fill in the blank using some property of the minimal polynomial of M or the characteristic polynomial of M. Prove that your statement is correct. The matrix M is similar to a diagonal matrix over F if and only if ________________ .

5. Let V be a finite dimension vector over the field F and let $T \colon V \to V$ be a linear transformation. Suppose that the characteristic polynomial of T is equal to $f_1^{e_1} f_2^{e_2} \dots f_r^{e_r}$, where the f_i are distinct irreducible polynomials in $F[X]$. For each i, let
$$W_i = \{v \in V \ : \ f_i^{e_i} v = 0\}.$$
Prove that $V = \oplus_{i=1}^r W_i$.

6. Suppose that M is a matrix with entries in the field F. Suppose further that
   - the characteristic polynomial of M is $(x-2)^{14}$,
   - the minimal polynomial of M is $(x-2)^4$, and
   - the dimension of the null space of $(M - 2I)$ is 5.

   Describe the steps you would take in order to determine the Jordan Canonical Form of M.

7. Give an example of a vector space V, a linear transformation $T \colon V \to V$, and a $T-$ invariant subspace W of V such that no complement of W in V is $T-$ invariant. Prove that your example does what it is supposed to do. (Note: The subspace W' of V is a complement of W in V if $W \oplus W' = V$.)

8. TRUE or FALSE. (If the statement is true, then prove it. If the statement is false, then give a counterexample. If you give a counterexample, then you must prove that your example does what it is supposed to do.) Let V be a finite dimensional vector space over the field of Complex Numbers. If $S\colon V \to V$ and $T\colon V \to V$ are diagonalizable linear transformations, then there exists a basis $\mathcal{B}$ for V such that the matrix of T with respect to $\mathcal{B}$ and the matrix of S with respect to $\mathcal{B}$ are both diagonal.

9. TRUE or FALSE. (If the statement is true, then prove it. If the statement is false, then give a counterexample. If you give a counterexample, then you must prove that your example does what it is supposed to do.) Let $(V, < >)$ be a finite dimensional Inner Product Space over the field of Complex Numbers. If $T\colon V \to V$ is a linear transformation, and $T(v) = cv$ for some vector $v \in V$ and some scalar $c \in \mathbb{C}$, then $T^{\mathrm{Adj}}(v) = \bar{c}v$.