MATH 700
HOMEWORK 8
DUE FRIDAY OCTOBER 20, 1989
AT THE BEGINNING OF CLASS.

1. Suppose that V is a vector space over a field F and K is a field which contains F. A typical element of $V \otimes_F K$ looks like $\sum_{\text{finite}} v_i \otimes k_i$, where $v_i \in V$ and $k_i \in K$. Furthermore, these elements are subject to the rules:

$$(v + v') \otimes k = (v \otimes k) + (v' \otimes k)$$
$$v \otimes (k + k') = (v \otimes k) + (v \otimes k')$$
$$fv \otimes k = v \otimes fk$$

for all $v, v' \in V$; $k, k' \in K$; and $f \in F$.

Now I will state the problem. Adopt the data of the preceding paragraph. Assume that V is finite dimensional. Let W be any vector space over F. **Exhibit** an isomorphism

$$\varphi\colon \operatorname{Hom}_F(V, W) \otimes_F K \to \operatorname{Hom}_K(V \otimes_F K, W \otimes_F K).$$

In particular, if $\theta \in \operatorname{Hom}_F(V, W)$ and $k \in K$, then I want to know what $\varphi(\theta \otimes k)$ does to a typical element of of $V \otimes_F K$. I do not want the definition of the map φ to depend on any choice of basis. (Make sure you explain why φ is well-defined.) You are required to prove that φ is an isomorphism; you may want to use bases for this part of the argument. (Note: If I were doing this problem, I would use the Universal Mapping Property of tensor products many times.)

2. Give an example of two nilpotent 4×4 matrices that have the same minimal polynomial, but are not similar. (Note: The matrix A is *nilpotent* if $A^r = 0$ for some integer r. The matrices A and B are *similar* if there is an invertible matrix M so that $A = M^{-1}BM$. In other words, the matrices A and B are similar if there is a linear transformation T so that A represents T with respect to one basis and B represents T with respect to some other basis.)

3. Let M be an $n \times m$ matrix over a field F. Prove that the dimension of the row space of M is equal to the dimension of the column space of M. (Note: I want a self-contained proof which does not quote a bunch of Lemmas from a book.)

4. (Brown. Page 130, number 4.) Let

$$A = \begin{bmatrix} 18 & -19 & -6 \\ 17 & -9 & -5 \\ 25 & -12 & -9 \end{bmatrix}$$

be a matrix over $\mathbb{Q}$. Show that A is nilpotent. Find an invertible matrix P such that PAP^{-1} is in Jordan Canonical form.