

Exam 1, 1989

There are 6 problems worth a total of 100 points.

- (15 points) Give an example of a linear transformation $T: V \rightarrow V$ such that T is one-to-one, but T is not onto.
- (15 points) Let F be a field in which 2 is a unit, and let V be the vector space consisting of all $n \times n$ matrices over F . Let Sym be the subspace of V consisting of all symmetric matrices (i.e. $M^t = M$), and let Skew be the subspace of V consisting of all skew-symmetric matrices (i.e. $M^t = -M$). Prove that $V = \text{Sym} \oplus \text{Skew}$.
- (15 points) Let $T: V \rightarrow W$ be a linear transformation of finite dimensional vector spaces. Let α be a basis for V and β be a basis for W . Let M be the matrix of T with respect to the bases α and β . Let N be the matrix of T^* with respect to the dual bases of α and β .
 - How are M and N related?
 - Prove your assertion from (a).
- (15 points) State and prove the Cayley-Hamilton Theorem.
- (20 points) Let M be a square matrix with entries in the arbitrary field F . Fill in the blank using some property of the minimal polynomial of M or the characteristic polynomial of M . Prove that your statement is correct. The matrix M is similar to a diagonal matrix over F if and only if _____.
- (20 points) Let M be a square matrix with complex entries. Suppose that M is in Jordan canonical form. A quick glance at M should tell you all of the eigenvalues of M and the dimension of each eigenspace. (Recall that the scalar a is an *eigenvalue* of M if $Mv = av$ for some non-zero vector v . The *eigenspace* associated to the eigenvalue a is $\{v \mid Mv = av\}$.)
 - What are the eigenvalues of M .
 - What is the dimension of each eigenspace?
 - Prove that your answers to (a) and (b) are correct.