Math 700 Fall 2003 Final Exam Solutions

Note! Write your answers on the blank sheets of paper provided. Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

Each problem is worth five points.

1. **Let V be a subspace of $\mathbb{R}^n$ and $W = \{w \in \mathbb{R}^n \mid w^{\mathrm{T}}v = 0$ for all $v \in V\}$. Prove that $\mathbb{R}^n = V + W$ and $V \cap W = \{0\}$.**

Intersection: Suppose $v \in V \cap W$, then $v^{\mathrm{T}}v = 0$. But $v^{\mathrm{T}}v$ is a sum of perfect squares in $\mathbb{R}$. Such a sum is zero only if each summand is zero. Thus, $v = 0$.
Sum: By induction find a basis $v_1, \dots, v_r$ for V so that $v_i^{\mathrm{T}}v_j = 0$ provided $i \neq j$. (If $v_1, \dots, v_p$ have already been found, and $p < \dim V$, then let v be any element of V which is not in the span of $v_1, \dots, v_p$. Let $v_{p+1} = v - \sum_{i=1}^{p} \frac{v_i^{\mathrm{T}}v}{v_i^{\mathrm{T}}v_i}v_i$.)
Let u be an arbitrary element of $\mathbb{R}^n$. Observe that

$$u = \left(u - \sum_{i=1}^{r} \frac{v_i^{\mathrm{T}}u}{v_i^{\mathrm{T}}v_i}v_i\right) + \sum_{i=1}^{r} \frac{v_i^{\mathrm{T}}u}{v_i^{\mathrm{T}}v_i}v_i,$$

where $\sum_{i=1}^{r} \frac{v_i^{\mathrm{T}}u}{v_i^{\mathrm{T}}v_i}v_i \in V$ and $u - \sum_{i=1}^{r} \frac{v_i^{\mathrm{T}}u}{v_i^{\mathrm{T}}v_i}v_i \in W$.

2. **State and prove the Cayley-Hamilton Theorem.**

If A is an $n \times n$ matrix with entries from the field F, and the polynomial $c(x)$ is defined to be the determinant of $xI - A$, then the matrix $c(A)$ is equal to zero.

Proof. On the last homework problem you did for me, you proved that the following sequence of $F[x]$-module homomorphisms is exact:

$$F[x]^n \xrightarrow{g} F[x]^n \xrightarrow{f} F^n \to 0,$$

where $f\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = \sum_{i=1}^{n} r_i e_i$, for the standard basis $e_1, \dots, e_n$ of F^n, and g is multiplication by the matrix $xI - A$. (As always, the vector space F^n is an $F[x]$-module by way of the action $x \cdot v = Av$.) Notice that when g is applied to column j of the classical adjoint of $xI - A$, then the result is the column vector whose only non-zero entry is $c(x)$ in position j. This column vector is in the kernel of f; hence, $0 = c(x) \cdot e_j = c(A)e_j$, and this is column j of $c(A)$. Repeat for each j to see that $c(A)$ is identically zero. (By the way, this argument only used that $\operatorname{im} g \subseteq \ker f$ (that is the direction that everyone got). This argument did not require the direction $\ker f \subseteq \operatorname{im} g$.) $\square$

3. **Exhibit a matrix B (with real entries) such that $B^2 = A$ for $A = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix}$.**

A quick doodle shows you that $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$ and $A\begin{bmatrix}1\\-1\end{bmatrix} = 4\begin{bmatrix}1\\-1\end{bmatrix}$. It follows that

$$A\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 4\end{bmatrix}.$$

In other words,

$$A = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}.$$

One good choice for B is

$$B = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}3 & -1\\-1 & 3\end{bmatrix}.$$

4. **Let $T\colon V \to V$ be a linear transformation of a finite dimensional vector space over the field F. Suppose that V_1 and V_2 are T-cyclic subspace of V and that the minimal polynomials of $T|_{V_1}$ and $T|_{V_2}$ are relatively prime. Prove that $V_1 + V_2$ is T-cyclic. (I expect a complete self-contained proof of this elementary Lemma. Do not appeal to the canonical form theorems or state that we did this in class.)**

Let v_i be a T-cyclic generator for V_i. Let g_i be the minimal polynomal of T acting on v_i. The hypothesis ensures that there exist polynomials A and B with $Ag_1 + Bg_2 = 1$. Let $v = v_1 + v_2$. I will show that v generates all of V as a T-module. Notice that $[A(T)g_1(T)](v) = [A(T)g_1(T)](v_1) + [I - B(T)g_2(T)]v_2 = v_2$. Similarly, $[A(T)g_2(T)](v) = v_1$.

5. **Let $T\colon V \to W$ be a linear transformation of finite dimensional vector spaces. State the formula which relates the dimension of the null space of T and the dimension of the image of T. Prove the formula you have stated.**

The rank of T plus the nullity of T is equal to the dimension of V.

Proof. Let $w_1, \dots, w_r$ in W be a basis for the image of T. Let $v_1, \dots, v_r$ be vectors in V with $T(v_i) = w_i$. Let $v_1', \dots, v_s'$, in V, be a basis for the kernel of T. I will show that $v_1, \dots, v_r, v_1', \dots, v_s'$ is a basis for V.

Span If $v \in V$, then $T(v)$ is in the image of T; so, $T(v) = \sum_{i=1}^{r} c_i w_i$ for some constants c_i. So, $v - \sum_{i=1}^{r} c_i v_i$ is in the kernel of T and can be written as a linear combination of the v_i'.

Linear Independence If $\sum_{i=1}^{r} c_i v_i + \sum_{j=1}^{s} c_j' v_j' = 0$, then apply T to see that $\sum_{i=1}^{r} c_i w_i = 0$; so each $c_i = 0$. The v_j' are linearly independent to the c_j' are also zero. □

6. **Let A and B be $n \times n$ matrices over $\mathbb{C}$ with $AB = BA$. Prove that A and B have a common eigenvector.**

The characteristic polynomial of A factors into linear factors over $\mathbb{C}$; so A has at least one eigenvalue λ. Let V be the subspace of eigenvectors of A which belong to λ. Observe that V is B-invariant since if $v \in V$, then $ABv = BAv = \lambda Bv$, and Bv is also an eigenvector of A which belongs to λ. Multiplication by B is a linear transformation on the Complex vector space V. The characteristic polynomial of this linear transformation also factors into linear factors. Hence, B has an eigenvector in V.