Math 700 Fall 2003 Final Exam

Note! Write your answers on the blank sheets of paper provided. Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

Each problem is worth five points.

1. Let V be a subspace of $\mathbb{R}^n$ and $W = \{w \in \mathbb{R}^n \mid w^{\mathrm{T}}v = 0 \text{ for all } v \in V\}$. Prove that $\mathbb{R}^n = V + W$ and $V \cap W = \{0\}$.

2. State and prove the Cayley-Hamilton Theorem.

3. Exhibit a matrix B (with real entries) such that $B^2 = A$ for $A = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix}$.

4. Let $T\colon V \to V$ be a linear transformation of a finite dimensional vector space over the field F. Suppose that V_1 and V_2 are T-cyclic subspace of V and that the minimal polynomials of $T|_{V_1}$ and $T|_{V_2}$ are relatively prime. Prove that $V_1 + V_2$ is T-cyclic. (I expect a complete self-contained proof of this elementary Lemma. Do not appeal to the canonical form theorems or state that we did this in class.)

5. Let $T\colon V \to W$ be a linear transformation of finite dimensional vector spaces. State the formula which relates the dimension of the null space of T and the dimension of the image of T. Prove the formula you have stated.

6. Let A and B be $n \times n$ matrices over $\mathbb{C}$ with $AB = BA$. Prove that A and B have a common eigenvector.