

Math 700 Fall 2003 Exam 1

Note! Write your answers on the blank sheets of paper provided. Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

Each problem is worth five points.

1. Let V be a vector space and S and T be linear transformations from V to V . Suppose $S \circ T$ is equal to the identity map. Answer each question. If your answer is YES, then **prove** the result. If your answer is NO, then give a **counterexample**.
 - (a) If the dimension of V is arbitrary, then must S be onto?
 - (b) If the dimension of V is arbitrary, then must S be one-to-one?
 - (c) If the dimension of V is finite, then must S be onto?
 - (d) If the dimension of V is finite, then must S be one-to-one?
2. Let V be a finite dimensional vector space over the field F and let S and T be linear transformations from V to V . Fix a basis for V . Let $[S]$, $[T]$, and $[S \circ T]$ be the matrices for S , T , and $S \circ T$ with respect to the fixed basis of V .
 - (a) What is the relationship between the three matrices?
 - (b) Prove your answer to (a). Justify each step carefully. (You might want to write down some definitions or some lemmas in order to make your argument coherent.)
3. True or False. If true, **prove** it. If false, then give a **counterexample**. If W_1 , W_2 , and W_3 are subspaces of the vector space V , then

$$W_1 \cap (W_2 + W_3) = (W_1 \cap W_2) + (W_1 \cap W_3).$$

4. Let V be a vector space of arbitrary dimension over the field F ; let B be a basis for V ; and let S be a linearly independent subset of V . Prove that there exists a subset S_1 of B such that $S \cup S_1$ is a basis for V .
5. Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation which is given by reflection across the plane $x + 2y + 3z = 0$. What is the matrix for T with respect to the standard basis of \mathbb{R}^3 ?