MATH 700
HOMEWORK 8

Due Friday, October 31, 2003 at the beginning of class.

Question 1. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation which is given by multiplication by the matrix

$$A = \frac{1}{4}\begin{bmatrix} 6 & 1 & 0 & -1 \\ -2 & 2 & -2 & 0 \\ 0 & 1 & 6 & 1 \\ 2 & 0 & -2 & 2 \end{bmatrix}.$$

Find a basis $\mathcal{B}$ for $\mathbb{R}^4$ so that the matrix of T with respect to $\mathcal{B}$ is

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Answer. Let $\mathcal{B}$ be the vectors v_1, v_2, v_3, v_4. We see that v_2 and v_4 are eigenvectors of A which belong to the eigenvalue 1. We find these vectors first. We solve $(A - I)v = 0$. Of course, this is equivalent to solving $(4A - 4I)v = 0$.

$$4A - 4I = \begin{bmatrix} 2 & 1 & 0 & -1 \\ -2 & -2 & -2 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & 0 & -2 & -2 \end{bmatrix}$$

Replace row 2 by row 2 plus row 1. Replace row 4 by row 4 minus row 1. Obtain

$$\begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & -1 & -2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & -1 \end{bmatrix}.$$

Replace row 1 by row 1 plus row 2; row 3 by row 3 plus row 2; and row 4 by row 4 minus row 2 to obtain

$$\begin{bmatrix} 2 & 0 & -2 & -2 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Divide row 1 by 2 and multiply row 2 by -1 to obtain

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is now easy to solve $(A - I)v = 0$. We take

$$v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors v_1 and v_3 satisfy $Av_1 = v_1 + v_2$ and $Av_3 = v_3 + v_4$. We solve $(A - I)v_1 = v_2$ and $(A - I)v_3 = v_4$. I already showed you that I know how to do row operations, so I will just show you my answers here. One solution is

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_3 = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

The vectors $\mathcal{B} = \{v_1, v_2, v_3, v_4\}$ are obviously linearly independent. It is also clear that $T(v_1) = v_1 + v_2$, $T(v_2) = v_2$, $T(v_3) = v_3 + v_4$, and $T(v_4) = v_4$. Thus, the matrix of T, with respect to $\mathcal{B}$ is given by M.

Question 2. Let $r_1, r_2, \ldots, r_n$ be elements of the commutative ring R. What is the determinant of the $n \times n$ matrix

$$M = \begin{bmatrix} 1 & \ldots & 1 \\ r_1 & \ldots & r_n \\ \vdots & & \vdots \\ r_1^{n-1} & \ldots & r_n^{n-1} \end{bmatrix}?$$

Prove your answer.

Answer. The matrix M is called a Vandermonde matrix. It is well known that the determinant is $\prod\limits_{i<j} (r_j - r_i)$. One proof goes by induction on n. The result is true when $n = 2$. Now we assume $3 \le n$. The determinant is unchanged if we perform elementary row operations to M. Replace row n by row n minus r_1 times row $n-1$. Replace row $n-1$ by row $n-1$ minus r_1 times row $n-2$. $\ldots$ Replace row 2 by row 2 minus r_1 times row 1. Thus,

$$\det M = \det \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 0 & r_2 - r_1 & \ldots & r_n - r_1 \\ 0 & r_2^2 - r_1 r_2 & \ldots & r_n^2 - r_1 r_n \\ 0 & r_2^3 - r_1 r_2^2 & \ldots & r_n^3 - r_1 r_n^2 \\ \vdots & \vdots & & \vdots \\ 0 & r_2^{n-1} - r_1 r_2^{n-2} & \ldots & r_n^{n-1} - r_1 r_n^{n-2} \end{bmatrix}$$

$$= \det \begin{bmatrix} r_2 - r_1 & \ldots & r_n - r_1 \\ r_2^2 - r_1 r_2 & \ldots & r_n^2 - r_1 r_n \\ r_2^3 - r_1 r_2^2 & \ldots & r_n^3 - r_1 r_n^2 \\ \vdots & & \vdots \\ r_2^{n-1} - r_1 r_2^{n-2} & \ldots & r_n^{n-1} - r_1 r_n^{n-2} \end{bmatrix}.$$

Pull $r_2 - r_1$ out of column 1; $r_3 - r_1$ out of column 2; etc. to see that

$$\det M = \prod_{j=2}^{n}(r_j - r_1)\det\begin{bmatrix} 1 & \dots & 1 \\ r_2 & \dots & r_n \\ \vdots & & \vdots \\ r_2^{n-2} & \dots & r_n^{n-2} \end{bmatrix}.$$

The claim is established by induction.