

**MATH 700**  
**HOMEWORK 7**

Due Friday, October 24, 2003 at the beginning of class.

1. Let  $0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3 \xrightarrow{T_3} 0$  be an exact sequence of linear transformations of vector spaces over the field  $F$ . (That is,  $\ker T_i = \text{im } T_{i-1}$  for all  $i$ .) Prove that

$$0 \xrightarrow{T_3^*} \text{Hom}_F(V_3, F) \xrightarrow{T_2^*} \text{Hom}_F(V_2, F) \xrightarrow{T_1^*} \text{Hom}_F(V_1, F) \xrightarrow{T_0^*} 0$$

is also an exact sequence of linear transformations of vector spaces. The vector spaces in this problem have arbitrary dimension.

**We show that  $T_2^*$  is one-to-one.** If  $\alpha \in \text{Hom}_F(V_3, F)$  and  $T_2^*(\alpha) = 0$ , then  $\alpha \circ T_2 = 0$ . The hypothesis tells us that  $T_2$  is onto. We conclude that  $\alpha$  is identically zero.

**We show that  $\text{im } T_2^* \subseteq \ker T_1^*$ .** Take  $\alpha \in \text{Hom}_F(V_3, F)$ . We know that  $T_1^*(T_2^*(\alpha)) = \alpha \circ T_2 \circ T_1$ . The hypothesis tells us that  $T_2 \circ T_1 = 0$ .

**We show that  $\ker T_1^* \subseteq \text{im } T_2^*$ .** Take  $\beta \in \text{Hom}_F(V_2, F)$  with  $T_1^*(\beta) = 0$ . In other words,  $\beta \circ T_1 = 0$ . We now define  $\alpha: V_3 \rightarrow F$ . Let  $\{v_i \mid i \in I\}$  be a basis for  $V_3$ . The linear transformation  $T_2$  is onto, so for each  $i \in I$ , there exists  $w_i \in V_2$  with  $T_2(w_i) = v_i$ . There is a unique linear transformation  $\alpha: V_3 \rightarrow F$  with  $\alpha(v_i) = \beta(w_i)$  for all  $i$ . Observe that  $T_2^*(\alpha) = \beta$  because  $\beta - \alpha \circ T_2$  takes every element of the image of  $T_1$  to zero;  $\beta - \alpha \circ T_2$  takes  $w_i$  to zero for all  $i \in I$ , and  $V_2$  is the sum of the image of  $T_1$  and the subspace of  $V_2$  which is spanned by  $\{w_i \mid i \in I\}$ .

**We show that  $T_1^*$  is onto.** Take  $\gamma \in \text{Hom}_F(V_1, F)$ . Let  $\{u_j \mid j \in J\}$  be a basis for  $V_1$ . The linear transformation  $T_1$  is one-to-one; hence,  $\{T_1(u_j) \mid j \in J\}$  is a linearly independent subset of  $V_2$ . Find vectors  $S$  in  $V_2$  so that  $S \cup \{T_1(u_j) \mid j \in J\}$  is a basis for  $V_2$ . Define a linear transformation  $\beta: V_2 \rightarrow F$  by  $\beta(T_1(u_j)) = \gamma(u_j)$  and  $\beta$  of each vector in  $S$  is zero. Observe that  $T_1^*(\beta) = \gamma$ .

2. Let  $R$  be a commutative ring. Let  $I, J$ , and  $K$  be ideals of  $R$  which satisfy  $I + J = R$ ,  $I + K = R$ , and  $J + K = R$ . Let  $\varphi: R \rightarrow \frac{R}{I} \oplus \frac{R}{J} \oplus \frac{R}{K}$  be the  $R$ -module homomorphism which sends  $r$  to  $(r+I, r+J, r+K)$ .
- (a) Prove that  $\varphi$  is onto.
- (b) Prove that the kernel of  $\varphi$  is  $IJK$ .

For (a), it suffices to show that  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$  are in the image of  $\varphi$ . By symmetry, it suffices to show that  $(1, 0, 0)$  is in the image of  $\varphi$ . I first claim that  $I + JK = R$ . The hypotheses ensures that there exist elements

$i, i' \in I$ ,  $j \in J$ , and  $k \in K$ , with  $i + j = 1$  and  $i' + k = 1$ . Multiply to get  $ii' + ik + ji' + jk = 1$ . We see that  $ii' + ik + ji' \in I$  and  $jk \in JK$ . The claim is established, we have  $i + y = 1$  for some  $i \in I$  and  $y \in JK$ . Notice that  $\varphi(y) = (1, 0, 0)$ . For (b), it is clear that the kernel of  $\varphi$  is  $I \cap J \cap K$ . It suffices to prove that  $I \cap J \cap K = IJK$ . It is clear that  $IJK \subseteq I \cap J \cap K$ . It suffices to prove  $I \cap J \cap K \subseteq IJK$ . We first prove that  $J \cap K \subseteq JK$ . The hypothesis guarantees the existence of elements  $j \in J$  and  $k \in K$  with  $j + k = 1$ . If  $x \in J \cap K$ , then  $x = xj + xk \in JK$ . Now take  $z \in I \cap J \cap K$ ; so,  $z \in I$  and  $z \in J \cap K \subseteq JK$ . We already found  $i + y = 1$  for some  $i \in I$  and  $y \in JK$ . Now we have  $z = zi + zy \in IJK$ .

3. Let

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

**Does there exist a vector  $v \in \mathbb{R}^3$  with  $v$ ,  $Mv$ , and  $M^2v$  a basis for  $\mathbb{R}^3$ ? Does there exist a vector  $v \in \mathbb{R}^3$  with  $v$ ,  $Nv$ , and  $N^2v$  a basis for  $\mathbb{R}^3$ ? When the answer is “yes”, then exhibit one such  $v$ . When the answer is “no”, then prove that such a  $v$  does not exist.**

Let  $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ . It is obvious that  $v$ ,  $Mv$ , and  $M^2v$  are linearly independent. It

is also obvious that  $w$ ,  $Nw$ , and  $N^2w$  are linearly dependent for all  $w \in \mathbb{R}^3$ .

For example, you can take  $w = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ .